

ABSTRACT

Title of dissertation: A VARIATIONAL APPROXIMATION SCHEME
FOR RADIAL POLYCONVEX ELASTICITY
THAT PRESERVES THE POSITIVITY
OF DETERMINANTS.

Alexey Miroshnikov, Doctor of Philosophy, 2012

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We study the equations describing the dynamics of radial motions for isotropic elastic materials; these form a system of non-homogeneous conservation laws. We construct a variational approximation scheme that decreases the total mechanical energy and at the same time leads to physically realizable motions that avoid interpenetration of matter.

In addition, we consider a variational scheme developed by S. Demoulini, D. Stuart and A. Tzavaras that approximates the equations of three-dimensional elastodynamics with polyconvex stored energy. We establish the convergence of the time-continuous interpolates constructed in the scheme to a solution of polyconvex elastodynamics before shock formation. The proof is based on a relative entropy estimation for the time-discrete approximants in an environment of L^p -theory bounds, and provides an error estimate for the approximation before the formation of shocks.

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POSITIVITY OF DETERMINANTS.

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2012

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Acknowledgments

Foremost, I would like to express my sincere gratitude to my research advisor, Dr. Athanasios Tzavaras, for his guidance, advice and encouragement during my doctoral studies. I strongly believe that without his immense knowledge, research experience and belief in my mathematical abilities, this thesis would not be possible.

I also would like to show my gratitude to Dr. Konstantina Trivisa, Dr. Stuart Antman and Dr. Manoussos Grillakis for their insightful comments and helpful scientific discussions that broadened my knowledge of the field and enabled me to analyze problems from different perspectives.

My sincere thanks also goes to Dr. Charalambos Makridakis, for offering me the research opportunities at the Institute of Applied Mathematics, FORTH in Crete, that lead to many exciting projects.

Last, but not least, I would like to thank my dear wife, Liudmila, for her enormous support, understanding and patience throughout the period of my studies.

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Chapter 1

Introduction

In continuum physics, material bodies are modeled as continuous media whose motion and equilibrium are governed by balance laws and constitutive relations. The list of balance laws identifies the theory, for instance, mechanics, thermomechanics, thermodynamics, etc., while the constitutive hypotheses describe the material response.

The equations describing the evolution of a continuous medium with nonlinear elastic response and zero body forces in referential description are given by

$$\frac{\partial^2 y}{\partial t^2} = \operatorname{div} S(\nabla y) \quad (1.1)$$

where $y : \Omega \times [0, \infty) \rightarrow \mathbb{R}^3$ stands for the motion, S for the first Piola-Kirchhoff stress tensor and the region Ω is the reference configuration of the elastic body. For hyperelastic materials the Piola-Kirchhoff stress is generated by a stored energy function,

$$S(F) = \frac{\partial W}{\partial F}(F), \quad (1.2)$$

an assumption which is motivated by considerations of thermodynamics. The equations (1.1) are often recast as a system of conservation laws,

$$\begin{aligned} \partial_t v_i &= \operatorname{div} S(F) \\ \partial_t F_{i\alpha} &= \partial_\alpha v_i, \end{aligned} \quad (1.3)$$

for the velocity $v = \partial_t y$ and the deformation gradient $F = \nabla y$. The equivalence of (1.1) and (1.3) holds for solutions (v, F) with F a gradient, $F = \nabla y$, a property equivalent to the set of differential constraints

$$\partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0 \tag{1.4}$$

The constraints (1.4) are an involution [10]: if they are satisfied at $t = 0$ then $(1.3)_2$ propagates (1.4) to hold for all times.

To avoid local interpenetration of matter it is natural to require that

$$\det \nabla y > 0 \tag{1.5}$$

or that y be locally invertible. To ensure that deformations satisfy (1.5) for a.e. $x \in \Omega$ it is often assumed that the stored energy

$$W(F) \rightarrow +\infty \quad \text{as} \quad \det F \rightarrow 0^+. \tag{1.6}$$

In addition, the requirement of frame indifference imposes that the stored energy $W(F)$ is to be invariant under rotations. This together with (1.6) renders the assumption of convexity of W too restrictive [31], and convexity has been replaced by various weaker conditions familiar from the theory of elastostatics, see [3, 5, 6]. A commonly employed assumption is that of polyconvexity, postulating that

$$W(F) = G \circ \Psi(F), \quad \Psi(F) := (F, \operatorname{cof} F, \det F)$$

with G convex; this encompasses certain physically realistic models [8, Sec 4.9, 4.10]. Starting with the work of Ball [3], substantial progress has been achieved for handling the lack of convexity of W within the existence theory of elastostatics.

Nevertheless there are several challenging problems that remain open. For instance, it has not yet been shown that the minimizer of

$$J[y] = \int_{\Omega} W(\nabla y) \, dx$$

is even a weak solution of the Euler-Lagrange equations (which are equations of elastostatics) in the case when the stored energy becomes infinite as $\det F \rightarrow 0^+$.

This is a challenging and difficult problem.

For the elastodynamics system local existence of classical solutions has been established in [12], [11, Thm 5.4.4] for rank-1 convex stored energies, and in [11, Thm 5.5.3] for polyconvex stored entropies. The existence of global weak solutions is an open problem, except in one-space dimension, see [17]. Construction of entropic measure-valued solutions has been achieved in [15] using a variational approximation method associated to a time-discretized scheme. Various uniqueness results of smooth solutions in the class of entropy weak and even dissipative measure valued solutions are available for the elasticity system [10, 11, 15, 21].

There are two interrelated objectives of the present work. The first one is to show that the approximation scheme proposed by S. Demoulini, D. Stuart and A. Tzavaras [15] converges to the classical solution of the elastodynamics system (1.1) before the formation of shocks [23]. Since the scheme in [15] does not take into account the constraint of positive determinant necessary to interpret y as a physically realizable motion, our second objective is to devise a variational scheme [22] that preserves the positivity of determinants (1.5).

Problem 1. A variational approximation method based on the time-discretization of

the extended elasticity system and designed to handle (spatially) periodic solutions to (1.3) (defined on the torus $\Omega := \mathbb{T}^3$) has been proposed in [15]: Given a time-step $h > 0$ and initial data (v^0, Ξ^0) the scheme provides the sequence of iterates (v^j, Ξ^j) , $j \geq 1$, by solving

$$\begin{aligned} \frac{v_i^j - v_i^{j-1}}{h} &= \partial_\alpha \left(\frac{\partial G}{\partial \Xi^A}(\Xi^j) \frac{\partial \Psi^A}{\partial F_{i\alpha}}(F^{j-1}) \right) \\ \frac{(\Xi^j - \Xi^{j-1})_A}{h} &= \partial_\alpha \left(\frac{\partial \Psi^A}{\partial F_{i\alpha}}(F^{j-1}) v_i^j \right) \end{aligned} \quad \text{in } \mathcal{D}'(\Omega). \quad (1.7)$$

Spatial iterates $v^j, \Xi^j = (F^j, Z^j, w^j)$ approximate the velocity $v = \partial_t y$ and the vector of null-Lagrangians $\Psi(\nabla y) = (\nabla y, \text{cof } \nabla y, \det \nabla y)$, respectively, at time $t = t_j$. This problem is solvable using variational methods and the iterates (v^j, Ξ^j) give rise to a time-continuous approximate solution $\Theta^{(h)} = (V^{(h)}, \Xi^{(h)})$. It has been shown in [15] that the approximate solution generates a measure-valued solution of the equations of polyconvex elastodynamics.

In this work we consider a smooth solution of the extended elasticity system $\bar{\Theta} = (\bar{V}, \bar{\Xi})$ defined on $\Omega \times [0, T]$ and show that the approximate solution $\Theta^{(h)}$ constructed via the iterates (v^j, Ξ^j) of (1.7) converges to $\bar{\Theta} = (\bar{V}, \bar{\Xi})$ at a convergence rate $O(h)$. The method of proof is based on the relative entropy method developed for convex entropies in [9, 16] and adapted for the system of polyconvex elasticity using the embedding of the system (1.1). The difference between $\Theta^{(h)}$ and $\bar{\Theta}$ is controlled by monitoring the evolution of the relative entropy

$$\eta^r = \frac{1}{2} |V^{(h)} - \bar{V}|^2 + G(\Xi^{(h)}) - G(\bar{\Xi}) - \nabla G(\bar{\Xi})(\Xi^{(h)} - \bar{\Xi})$$

We establish control of the function

$$\mathcal{E}(t) := \int_{\Omega} \left((1 + |F^{(h)}|^{p-2} + |\bar{F}|^{p-2}) |F^{(h)} - \bar{F}|^2 + |\Theta^{(h)} - \bar{\Theta}|^2 \right) dx.$$

and prove the estimation

$$\mathcal{E}(\tau) \leq C \left(\mathcal{E}(0) + h \right), \quad \tau \in [0, T],$$

which provides the result. There are two novelties in the present work: (a) Adapting the relative entropy method to the subject of time-discretized approximations. (b) Employing the method in an environment where L^p -theory needs to be used for estimating the relative entropy.

Problem 2. The aforementioned scheme is designed to approximate the map $y(x, t)$ that solves (1.1) and has (spatially) periodic $v = \partial_t y$ and $F = \nabla y$. However, if the solution y to (1.1) does not satisfy (1.5) then strictly speaking it may not be interpreted as an elastic motion. One of the shortcomings of the scheme developed in [15] is that the condition (1.5) does not, in general, hold for the approximants.

To address this issue we consider the equations describing *radial* motions of nonlinear, isotropic, elastic materials

$$w_{tt} = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial \Phi}{\partial v_1} \left(w_R, \frac{w}{R}, \frac{w}{R} \right) \right) - \frac{1}{R} \sum_{i=2}^3 \frac{\partial \Phi}{\partial v_i} \left(w_R, \frac{w}{R}, \frac{w}{R} \right). \quad (1.8)$$

Here, y stands for a radial motion $y(x, t) = w(R, t) \frac{x}{R}$, $R = |x|$, the stored energy (due to isotropy) is expressed as a function $W(F) = \Phi(v_1, v_2, v_3)$ of principal stretches of F and (1.8) monitors the evolution of its amplitude $w(R, t)$. In the radial case, a necessary condition (1.5) for y to represent a physically realizable motion dictates

$$w_R (w/R)^2 > 0, \quad (1.9)$$

and is also a sufficient condition for avoiding interpenetration of matter. To incorporate (1.9) into the variational scheme we employ a polyconvex stored energy

$$\begin{aligned} W(F) &= \Phi(v_1, v_2, v_3) =: G(v_1, v_2, v_3, v_2v_3, v_1v_3, v_1v_2, v_1v_2v_3) \\ &:= \phi(v_1) + \phi(v_2) + \phi(v_3) + g(v_2v_3) + g(v_1v_3) + g(v_1v_2) + h(v_1v_2v_3), \end{aligned}$$

where ϕ , g and h are convex functions and $h(\delta) \rightarrow +\infty$ as $\delta \rightarrow 0+$.

In the present work, we consider the equations (1.8) and devise a variational approximation scheme that on one hand preserves the positivity of determinants (1.9) and on the other produces a time-discretized variant of entropy dissipation. Similar to [15], the scheme employs transport identities for the null-Lagrangians – potential energies $\Psi(v_1, v_2, v_3; R)$ for which the functional

$$I[w] = \int_0^1 \Psi((w_R, \frac{w}{R}, \frac{w}{R}); R) dR$$

has variational derivative zero. In our case, Ψ 's are computed to be the functions v_1 , v_1v_2R , v_1v_3R or $v_1v_2v_3R^2$. Along solutions of the dynamical problem, each null-Lagrangian satisfies the transport identity $\partial_t \Psi = \partial_R(\frac{\partial \Psi}{\partial v_1} v)$ with Ψ and $\Psi_{,i}$ evaluated at $(w_R, w/R, w/R; R)$. These identities allow to embed system (1.8) into the symmetrizable first-order evolution system. In addition, we make a change of variables suggested in Ball [4] (for the equilibrium problem) setting $\rho = R^3$, $\alpha = w^3$, $\beta = w_R/R^2$, $\gamma = w^2$, and $v = w_t$. The essence of the transformation is to consider the antiderivative of the determinant $w_R(w/R)^2 = \alpha_\rho$ as the prime variable in the minimization problem, in conjunction with the null-Lagrangian transport identities when expressed for the new variable. In the end the extended system has four actual unknowns α , β , γ and v and is the symmetrizable system endowed with the convex

entropy.

The method we develop is based on the time discretization of the extended system. In fact, the equations of the time-discretized extended system are the Euler-Lagrange equations associated with the following variational problem: given $(v_j, \alpha_j, \beta_j, \gamma_j)$ minimize

$$I(\alpha, \beta, \gamma, v) = \int_0^1 \frac{1}{2}(v - v_j)^2 + G(\Xi) d\rho, \quad (1.10)$$

where

$$\Xi = \left(\beta \rho^{2/3}, \frac{\alpha}{\rho}, \frac{\alpha}{\rho}, \frac{\gamma}{\rho^{1/3}}, \frac{3\gamma_\rho}{2} \rho^{2/3}, \frac{3\gamma_\rho}{2} \rho^{2/3}, \alpha_\rho \rho^{2/3} \right) \in \mathbb{R}^7,$$

over the set of admissible functions

$$\begin{aligned} \mathcal{A}_\lambda = \left\{ (\alpha, \beta, \gamma, v) \in X : \alpha(0) \geq 0, \alpha(1) = \lambda, \alpha' > 0 \text{ a.e.,} \right. \\ \left. I(\alpha, \beta, \gamma, v) < \infty \text{ and } \frac{(\beta - \beta_j)}{h} = 3v', \right. \\ \left. \frac{(\alpha - \alpha_j)}{h} = 3\alpha_j^{2/3}v, \frac{(\gamma - \gamma_j)}{h} = 2\alpha_j^{1/3}v \right\}. \end{aligned} \quad (1.11)$$

The differential constraints in (1.11) are affine, the condition $\alpha(1) = \lambda$ corresponds to the imposed boundary condition $w(1, t) = \lambda$, while $\alpha' > 0$ ensures the positivity of determinants (1.9). We prove the existence and uniqueness of a minimizer for the functional I over \mathcal{A}_λ and show that the minimizer is a weak solution to the corresponding Euler-Lagrange equations. The analysis of the minimization problem (1.10)-(1.11) uses direct methods of the calculus of variations, in the spirit of [4], with the novel element of accounting for the evolutionary constraints in (1.11).

Thesis organization. In *Chapter 2* we consider *Problem 1* and cover convergence of the three dimensional variational scheme. The chapter is split into two sections.

Section 2.1 introduces the equations of elasticity along with the basic physical requirements and constitutive hypotheses on the stored energy. It also presents the variational scheme [15] developed by S. Demoulini, D. Stuart and A. Tzavaras. In Section 2.2 we introduce the notion of relative entropy, derive a relative entropy identity and finally prove convergence of the variational scheme.

In *Chapter 3* we deal with *Problem 2*. The chapter is split into three sections. Section 3.1 introduces the equations of radial elasticity for isotropic materials. Section 3.2 introduces null-Lagrangians in the radial case and presents two possible symmetrizable extensions to radial elastodynamics. Finally, in Section 3.3 we develop a variational scheme that decreases the total mechanical energy and at the same time leads to physically realizable radial motions that avoid interpenetration of matter.

Chapter 2

Convergence of Variational Schemes for Elastodynamics with Polyconvex Energy

The purpose of this chapter is to present a variational scheme developed in [15] that approximates the equations of three-dimensional elastodynamics with polyconvex stored energy and then establish the convergence of the time-continuous interpolates constructed in the scheme to a solution of polyconvex elastodynamics before shock formation [23].

2.1 Background Information

In this section we present the equations of nonlinear elasticity, introduce the notion of stored energy and that of entropy-entropy flux pair as well as discuss physical realizability of elastic motions.

2.1.1 Hyperelastic Elastodynamics

The equations of nonlinear elasticity (with zero body forces) are the system

$$\frac{\partial^2 y}{\partial t^2} = \operatorname{div} S(\nabla y) \tag{2.1}$$

where $y : \Omega \times [0, \infty) \rightarrow \mathbb{R}^3$ stands for the elastic motion, S for the Piola-Kirchhoff stress tensor and the region Ω is the reference configuration of the elastic body.

The equations (2.1) are often recast as a system of conservation laws,

$$\partial_t v_i = \partial_\alpha S_{i\alpha}(F) \quad (2.2)$$

$$\partial_t F_{i\alpha} = \partial_\alpha v_i,$$

for the velocity $v = \partial_t y$ and the deformation gradient $F = \nabla y$. The differential constraints

$$\partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0 \quad (2.3)$$

are propagated from the kinematic equation (2.2)₂ and are an involution [10]: if they are satisfied for $t = 0$ then (2.2) propagates (2.3) to satisfy for all times. Thus the system (2.2) is equivalent to systems (2.1) whenever $F(\cdot, 0)$ is a gradient.

In our work we employ the constitutive theory of *hyperelasticity* in which case the first Piola-Kirchhoff stress tensor S is expressed as the gradient,

$$S(F) = \frac{\partial W}{\partial F}(F), \quad (2.4)$$

of the stored-energy function of the elastic body

$$W : M^{3 \times 3} \rightarrow \mathbb{R}$$

where $M^{3 \times 3}$ is the set of real 3×3 matrices.

Physical realizability of elastic motions. In order for the geometric mapping $y(x, t) : \Omega \times [0, \infty) \rightarrow \mathbb{R}^3$ to correspond to a physically realizable *motion* one has to exclude interpenetration of matter. As a minimum requirement the condition

$$\det \nabla y > 0 \quad \text{a.e. } x \in \Omega \quad (2.5)$$

is often imposed which ensures that compression of a finite volume down to a point would not occur or that the map y be injective.

In addition, the stored energy function W must satisfy the physical requirement of *frame-indifference*, i.e., for all proper rotations $Q \in SO(3)$

$$W(F) = W(FQ), \quad \forall F \in M^{3 \times 3}. \quad (2.6)$$

Hyperbolicity. Consider the system of balance laws

$$\partial_t U(x, t) + \operatorname{div} G(U(x, t), x, t) = \Pi(U(x, t), x, t) \quad (2.7)$$

where $U(x, t) : \mathcal{X} \subset \mathbb{R}^m \times \mathbb{R} \rightarrow \mathcal{O} \subset \mathbb{R}^n$ and \mathcal{X}, \mathcal{O} are open sets.

Definition 2.1. The system of balance laws (2.7) is called *hyperbolic* in the t -direction if, for any fixed $U \in \mathcal{O}$, $(x, t) \in \mathcal{X}$ and $\nu \in S^{m-1}$, the eigenvalue problem

$$\left[\sum_{\alpha=1}^m DG_{\alpha}(U, x, t) - \lambda I \right] R = 0 \quad (2.8)$$

has real eigenvalues $\lambda_1(\nu; U, x, t), \dots, \lambda_n(\nu; U, x, t)$, called *characteristic speeds*, and n linearly independent eigenvectors $R_1(\nu; U, x, t), \dots, R_n(\nu; U, x, t)$.

Remark. In the above definition, D stands for the differential with respect to the U variable and denotes $[\partial/\partial U^1, \dots, \partial/\partial U^n]$, regarded as a row operation.

Let \mathcal{F} be a smooth m -dimensional manifold, embedded in the open subset $\mathcal{X} \subset \mathbb{R}^m \times \mathbb{R}$, with orientation induced by the unit normal field N . Assume that a measurable field U is a weak solution of the system of balance laws (2.7) on \mathcal{X} , i.e., $G(U(x, t), x, t)$ and $\Pi(U(x, t), x, t)$ are locally integrable and

$$\int_{\mathcal{X}} [G(U(x, t), x, t) \operatorname{grad} \varphi(x, t) + \varphi(x, t) \Pi(U(x, t), x, t)] d(x, t) = 0 \quad (2.9)$$

for any test function $\varphi \in C_0^\infty(\mathcal{X})$. Finally, assume that U is continuously differentiable on $\mathcal{X} \setminus \bar{\mathcal{F}}$, but is allowed to be singular on \mathcal{F} . In particular, (2.7) holds for any $(x, t) \in \mathcal{X} \setminus \bar{\mathcal{F}}$.

The manifold \mathcal{F} is called a *weak front* if U is Lipschitz continuous on \mathcal{X} and as one approaches \mathcal{F} from either side the $\text{grad} U$ attains distinct limits $\text{grad}^- U$, $\text{grad}^+ U$. In this case, $\text{grad} U$ has a jump

$$[[\text{grad} U]] = \text{grad}^+ U - \text{grad}^- U$$

across the manifold \mathcal{F} . Since U is continuous, tangential derivatives of U cannot jump across \mathcal{F} and hence $[[\text{grad} U]] = [[\partial U / \partial N]] \otimes N$, where $[[\partial U / \partial N]]$ denotes the jump of the normal derivative $\partial U / \partial N$ across \mathcal{F} . Therefore, taking the jump of (2.7) across \mathcal{F} at any point $(x, t) \in \mathcal{F}$ we get the following condition on the jump of the normal derivative [11]:

$$D[G(U(x, t)N)] \left[\left[\frac{\partial U}{\partial N} \right] \right] = 0 \quad (2.10)$$

where, as before, D denotes the differential with respect to U variable.

The definition of hyperbolicity maybe naturally interpreted in terms of the notion of weak fronts. If we renormalize the normal N on \mathcal{F} so that $N = (\nu, -s)$ with $\nu \in S^{m-1}$, then the wave will be propagating in the direction ν with speed s . Thus, comparing (2.8) with (2.10) we conclude that a system of n balance laws is hyperbolic if and only if n distinct waves can propagate in any spatial direction. The eigenvalues of (2.8) will determine the speed of propagation of these waves while the corresponding eigenvectors will specify the direction of their amplitude.

We turn our attention back to elastodynamics for hyperelastic materials. One

can verify [11, p. 55] that the first-order elasticity system (2.2) is hyperbolic on a certain region of the state space if for every F lying in the region the stored energy W satisfies *Legendre-Hadamard condition*

$$\frac{\partial^2 W(F)}{\partial F_{i\alpha} \partial F_{j\beta}} \nu_\alpha \nu_\beta \xi_i \xi_j > 0, \quad \text{for all } \nu \text{ and } \xi \text{ in } S^2 \quad (2.11)$$

which means that the stored energy W is *rank-one convex in F* , i.e., it is convex along any direction $\xi \otimes \nu$ with rank one.

An alternative way of expressing (2.11) is to state that for any unit vector ν the *acoustic tensor* $N(\nu, F)$, defined by

$$N_{ij}(\nu, F) = \frac{\partial^2 W(F)}{\partial F_{i\alpha} \partial F_{j\beta}} \nu_\alpha \nu_\beta, \quad i, j = 1, 2, 3 \quad (2.12)$$

is positive definite. In fact, for the elasticity system (2.2), the characteristic speeds are given by

$$\begin{aligned} \lambda_1 &= \cdots = \lambda_6 = 0, \\ \lambda_7 &= \cdots = \lambda_{12} = \pm \sqrt{\text{eigenvalues of the acoustic tensor}}. \end{aligned} \quad (2.13)$$

Entropy-entropy flux pairs. In continuum physics, weak solutions of a system of conservation laws, (2.7) with $\Pi \equiv 0$, are required to satisfy *entropy* inequalities of the form

$$\partial_t \eta(U(x, t), x, t) + \partial_\alpha q_\alpha(U(x, t), x, t) \leq 0 \quad (2.14)$$

where η, q , called *entropy-entropy flux pair*, are related by a first-order partial differential equation

$$Dq_\alpha(U, x, t) = D\eta(U, x, t), DG_\alpha(U, x, t), \quad U \in \mathcal{O}, (x, t) \in \mathcal{X}, \alpha = 1, \dots, m.$$

Such inequalities are a manifestation of irreversibility and as such originate from the second law of thermodynamics. For the system of (hyperelastic) elastodynamics an important *entropy-entropy flux pair* is

$$\eta(v, F) = \frac{1}{2}|v|^2 + W(F), \quad q_\alpha(v, F) = -S_{i\alpha}(F) v_i, \quad (2.15)$$

in which case (2.14) becomes

$$\partial_t \left(\frac{1}{2}|v|^2 + W(F) \right) - \partial_\alpha \left(v_i \frac{\partial W}{\partial F_{i\alpha}}(F) \right) \leq 0 \quad (2.16)$$

and expresses the *dissipation* of mechanical energy $\eta(v, F)$ on shocks. Notice that if (v, F) is a smooth solution to (2.2), after multiplying (2.2)₁ by v_i and then summing up over all indices $i = 1, 2, 3$, we find that (2.16) becomes equality which means that the mechanical energy of smooth solutions is conserved.

2.1.2 Polyconvex Stored Energy

Convexity of the stored energy is, in general, not a natural assumption since it is incompatible with certain physical requirements [2, Section 13.3]. Some of the reasons for rejecting convexity are presented below:

(i) One of the natural assumptions often imposed on the stored energy is that

$W(F) \rightarrow \infty$ as $\det F \rightarrow 0^+$ so that compression of a finite volume down to a point would cost infinite energy. Observe that the domain of W in this case is the open nonconvex set

$$M_+^{3 \times 3} := \{F \in M^{3 \times 3} : \det F > 0\}.$$

The above assumption is incompatible with the requirement of convexity for W since a convex W finite-valued on an open nonconvex set cannot approach infinity everywhere on the boundary of that set.

- (ii) Certain equilibrium problems admit nonunique solutions (e.g. buckled states) which would be prohibited by assuming strict convexity of the stored energy.
- (iii) Finally, strict convexity of the stored energy is, in general, incompatible with the requirement of frame-indifference. This is demonstrated by the following example. Suppose that W is smooth, strictly convex and frame-indifferent. Assume also that the material has a natural state, i.e., $S(I) = \frac{\partial W}{\partial F}(I) = 0$ which means that the body is free of stresses when in the reference configuration (for instance, $W(F) = |F|^2 - \text{tr}\{F\}$ for which $S(F) = F - I$ and hence $S(I) = 0$). Set $\bar{F} = I$ and $F = Q \in SO(3)$ such that $\bar{F} \neq F$. Then, (2.4) and the strict convexity of W imply

$$\begin{aligned} & [S(\bar{F}) - S(F)] : [\bar{F} - F] \\ &= \left[\int_0^1 \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}} (F + \tau(\bar{F} - F)) d\tau \right] (\bar{F} - F)_{i\alpha} (\bar{F} - F)_{j\beta} > 0. \end{aligned}$$

On the other hand, by (2.4), (2.6) and the assumption that $S(I) = 0$ we deduce

$$S(\bar{F}) - S(F) = S(I) - S(Q) = S(I) - QS(I) = 0$$

which contradicts to the above inequality.

In our work, as an alternative to convexity, we exploit the assumption of *polyconvexity* introduced by Ball [3]. It postulates that the stored energy W has the

form

$$W(F) = G \circ \Psi(F), \quad F \in M_+^{3 \times 3} \quad (2.17)$$

where

$$G = G(\Xi) = G(F, Z, w) : M_+^{3 \times 3} \times M^{3 \times 3} \times \mathbb{R} \cong \mathbb{R}^{19} \rightarrow \mathbb{R}$$

is a convex function and

$$\Psi(F) := (F, \operatorname{cof} F, \det F). \quad (2.18)$$

Motivation for polyconvexity. Consider an equilibrium problem for a hyperelastic material

$$0 = \operatorname{div} \left(\frac{\partial W}{\partial F}(\nabla y) \right) = \partial_{x_\alpha} \left(\frac{\partial W}{\partial F_{i\alpha}}(\nabla y) \right), \quad \alpha = 1, \dots, 3. \quad (2.19)$$

The equations (2.19) are the *Euler-Lagrange* equations associated with the functional

$$J[y] = \int_{\Omega} W(\nabla y(x)) \, dx. \quad (2.20)$$

To guarantee the existence of the minimizers it is essential [3] that the functional $J[\cdot]$ be sequentially weak* lower semicontinuous on $W^{1,\infty}(\Omega; M^{3 \times 3})$ (i.e. $y_k \xrightarrow{*} y$ in $W^{1,\infty}$ implies $J[y] \leq \lim_{k \rightarrow \infty} J[y_k]$). A suitable condition was introduced by Morrey [24] who showed that if W is *quasiconvex*, i.e. satisfies

$$\int_{\Omega} W(F + \nabla \varphi(x)) \, dx \geq W(F) |\Omega|, \quad \forall F \in M^{3 \times 3}, \varphi \in C_0^\infty(\Omega), \quad (2.21)$$

and certain growth hypothesis are satisfied, then the functional $J[\cdot]$ is weak* lower semicontinuous in $W^{1,\infty}$. However, the existence theorems of [24] fail to apply directly to nonlinear elasticity. The growth conditions used in [24] are too stringent; in particular, they prohibit the natural condition for W to increase without bound

as $\det F \rightarrow 0^+$. Ball [3] introduced the notion of polyconvexity (which is a particular case of quasiconvexity and capable of taking into account the physical requirements for the stored energy W) and established the corresponding existence theorem [3, Theorem 7.2] for the minimizer of (2.20) with polyconvex W . The proof hinges on the fact that maps $y \rightarrow \operatorname{cof} \nabla y : W^{1,p} \rightarrow L^{p/2}$ and $y \rightarrow \det \nabla y : W^{1,q} \rightarrow L^{q/3}$ are sequentially weakly continuous if $p > 2$ and $q > 3$ respectively.

Polyconvex elasticity. For polyconvex stored energies (2.17) the system of elastodynamics (2.1) is expressed by

$$\begin{aligned} \partial_t v_i &= \partial_\alpha \left(\frac{\partial G}{\partial \Xi_A}(\Psi(F)) \frac{\partial \Psi^A}{\partial F_{i\alpha}}(F) \right) \\ \partial_t F_{i\alpha} &= \partial_\alpha v_i \end{aligned} \tag{2.22}$$

which is equivalent to system (2.1) subject to differential constraints (2.3) that are an involution [10]. In addition, system (2.22) is endowed with the entropy identity

$$\partial_t \left(\frac{|v|^2}{2} + G(\Psi(F)) \right) - \partial_\alpha \left(v_i \frac{\partial G}{\partial \Xi_A}(\Psi(F)) \frac{\partial \Psi^A}{\partial F_{i\alpha}}(F) \right) = 0. \tag{2.23}$$

2.1.3 Variational Scheme in Three Dimensions

In this section we present the variational approximation method developed by Demoulini, Stuart and Tzavaras [15] that produces entropic measure valued solutions to the system of polyconvex elastodynamics (2.22) defined on the torus \mathbb{T}^3 . This scheme is the main subject of our investigation: Later in the sequel we will establish the convergence of the time-continuous interpolates constructed in the scheme to a solution of polyconvex elastodynamics before shock formation.

Extension to polyconvex elastodynamics. We first describe a symmetrizable extension of polyconvex elastodynamics introduced in [15] which is based on certain kinematic identities on $\text{cof } F$ and $\det F$ from [25].

Definition 2.2. *A continuous function $L(F) : M^{3 \times 3} \rightarrow \mathbb{R}$ is a null-Lagrangian if*

$$\int_{\Omega} L(\nabla(u + \varphi)(x)) dx = \int_{\Omega} L(\nabla u(x)) dx \quad (2.24)$$

for every bounded open set $\Omega \subset \mathbb{R}^3$ and for all $u \in C^1(\bar{\Omega}; \mathbb{R}^3)$, $\varphi \in C_0^\infty(\Omega; \mathbb{R}^3)$.

By making a suitable change of variables one can show that if (2.24) holds for some $\Omega = \Omega_0$ and for all u, φ then it holds for all Ω, u, φ . Take now arbitrary $\Omega, u \in C^1(\bar{\Omega}; \mathbb{R}^3)$, $\varphi \in C_0^\infty(\Omega; \mathbb{R}^3)$ and assume that $L \in C^1(M^{3 \times 3})$. Then

$$\frac{\partial}{\partial s} \int_{\Omega} L(\nabla(u + s\varphi)) dx = \sum_{i,\alpha=1}^3 \int_{\Omega} \frac{\partial L}{\partial F_{i\alpha}}(\nabla(u + s\varphi)) \frac{\partial \varphi^i}{\partial x_\alpha}(x) dx$$

and hence $L \in C^1(M^{3 \times 3})$ is a null-Lagrangian if and only if

$$\sum_{i,\alpha=1}^3 \int_{\Omega} \frac{\partial L}{\partial F_{i\alpha}}(\nabla u(x)) \frac{\partial \varphi^i}{\partial x_\alpha}(x) dx = 0$$

for all $\Omega, u \in C^1(\bar{\Omega}; \mathbb{R}^3)$, $\varphi \in C_0^\infty(\Omega; \mathbb{R}^3)$, i.e., if and only if the Euler-Lagrange equations

$$\sum_{i,\alpha=1}^3 \frac{\partial}{\partial x_\alpha} \left(\frac{\partial L}{\partial F_{i\alpha}}(\nabla u(x)) \right) = 0 \quad (2.25)$$

are identically satisfied in the sense of distributions for all $u \in C^1(\Omega; \mathbb{R}^3)$.

One can easily verify that the components of $\Psi(F)$ defined by (2.18) are null-Lagrangians [5] and hence (using the summation convention over repeated indices)

$$\partial_\alpha \left(\frac{\partial \Psi^A}{\partial F_{i\alpha}}(\nabla u) \right) = 0, \quad A = 1, \dots, 19 \quad (2.26)$$

for any smooth $u(x) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ or equivalently

$$\partial_\alpha \left(\frac{\partial \Psi^A}{\partial F_{i\alpha}}(F) \right) = 0, \quad \forall F \text{ with } \partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0.$$

The kinematic compatibility equation (2.22)₂ implies

$$\begin{aligned} \partial_t \Psi^A(F) &= \frac{\partial \Psi^A(F)}{\partial F_{i\alpha}} \partial_\alpha v_i \\ &= \partial_\alpha \left(\frac{\partial \Psi^A}{\partial F_{i\alpha}}(F) v_i \right) - v_i \partial_\alpha \left(\frac{\partial \Psi^A}{\partial F_{i\alpha}}(F) \right) \\ &= \partial_\alpha \left(\frac{\partial \Psi^A}{\partial F_{i\alpha}}(F) v_i \right), \quad \forall F \text{ with } \partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0. \end{aligned} \tag{2.27}$$

This motivates embed (2.22) into the system of conservation laws

$$\begin{aligned} \partial_t v_i &= \partial_\alpha \left(\frac{\partial G}{\partial \Xi_A}(\Xi) \frac{\partial \Psi^A}{\partial F_{i\alpha}}(F) \right) \\ \partial_t \Xi_A &= \partial_\alpha \left(\frac{\partial \Psi^A}{\partial F_{i\alpha}}(F) v_i \right). \end{aligned} \tag{2.28}$$

Note that $\Xi = (F, Z, w)$ takes values in $M^{3 \times 3} \times M^{3 \times 3} \times \mathbb{R} \simeq \mathbb{R}^{19}$ and is treated as a new dependent variable. Furthermore, observe that the components of F constitute the first nine components of Ξ and hence the equation (2.22)₂ is included as the first part of (2.28)₂.

The extension has the following properties:

- (E1) If $F(\cdot, 0)$ is a gradient then $F(\cdot, t)$ remains a gradient $\forall t$.
- (E2) If $F(\cdot, 0)$ is a gradient and $\Xi(\cdot, 0) = \Psi(F(\cdot, 0))$, then $F(\cdot, t)$ remains a gradient and $\Xi(\cdot, t) = \Psi(F(\cdot, t))$, $\forall t$. In other words, the system of polyconvex elastodynamics can be viewed as a constrained evolution of (2.28).

- (E3) The enlarged system admits a convex entropy

$$\eta(v, \Xi) = \frac{1}{2}|v|^2 + G(\Xi), \quad (v, \Xi) \in \mathbb{R}^{22} \tag{2.29}$$

and thus is symmetrizable (along the solutions that are gradients).

Assumptions [Demoulini-Stuart-Tzavaras]. In [15] the authors consider polyconvex stored energy W in the form $W(F) = G(\Psi(F))$ and work with periodic boundary conditions, *i.e.*, the spatial domain Ω is taken to be the three-dimensional torus \mathbb{T}^3 . The indices i, j, \dots generally run over $1, \dots, 3$ while A, B, \dots run over $1, \dots, 19$. The following notation is used: $L^p = L^p(\mathbb{T}^3)$, $L^\infty(L^p) = L^\infty((0, T); L^p(\mathbb{T}^3))$ and $Q_\infty = \mathbb{T}^3 \times [0, \infty)$. In addition, the following convexity and growth assumptions on G are posed:

(H1*) $G \in C^2(\text{Mat}^{3 \times 3} \times \text{Mat}^{3 \times 3} \times \mathbb{R}; [0, \infty))$ is a strictly convex function, *i.e.*,

$$\exists \gamma > 0 \text{ such that } D^2 G \geq \gamma > 0.$$

(H2*) $G(F, Z, w) \geq c_1|F|^p + c_2|Z|^q + c_3|w|^r - c_4$ where $p \in (4, \infty)$ and $q, r \in [2, \infty)$

are fixed.

(H3*) $G(F, Z, w) \leq c(|F|^p + |Z|^q + |w|^r + 1)$ with p, q, r as in (H2*).

(H4*) $|\partial_F G|^{\frac{p}{p-1}} + |\partial_Z G|^{\frac{p}{p-2}} + |\partial_w G|^{\frac{p}{p-3}} \leq C(|F|^p + |Z|^q + |w|^r + 1)$ with the exponents p, q, r as in (H2*).

Variational scheme. The variational approximation method proposed in [15] is based upon time-discretization of the extended system (2.28): Given initial data

$$\Theta^0 := (v^0, \Xi^0) = (v^0, F^0, Z^0, w^0) \in L^2 \times L^p \times L^q \times L^r$$

and fixed $h > 0$, the scheme constructs the sequence of successive iterates

$$\Theta^j := (v^j, \Xi^j) = (v^j, F^j, Z^j, w^j) \in L^2 \times L^p \times L^q \times L^r, \quad j \geq 1$$

that solve

$$\begin{aligned}\frac{v_i^j - v_i^{j-1}}{h} &= \partial_\alpha \left(\frac{\partial G}{\partial \Xi_A}(\Xi^j) \frac{\partial \Psi^A}{\partial F_{i\alpha}}(F^{j-1}) \right) \\ \frac{\Xi_A^j - \Xi_A^{j-1}}{h} &= \partial_\alpha \left(\frac{\partial \Psi^A}{\partial F_{i\alpha}}(F^{j-1}) v_i^j \right).\end{aligned}\quad \text{in } \mathcal{D}'(\mathbb{T}^3) \quad (2.30)$$

The existence of the iterates (v^j, Ξ^j) satisfying (2.30) is guaranteed by

Lemma 2.1 ([15], p. 333). *Given $(v^{j-1}, F^{j-1}, Z^{j-1}, w^{j-1}) \in L^2 \times L^p \times L^2 \times L^2$ there exists*

$$(v, \Xi) = (v, F, Z, w) \in L^2 \times L^p \times L^2 \times L^2$$

which minimizes the functional

$$\mathcal{J}(v, F, Z, w) = \int_{\mathbb{T}^3} \frac{1}{2} |v - v^{j-1}|^2 + G(F, Z, w) dx$$

on the weakly closed affine subspace

$$\begin{aligned}\mathcal{C} = \Big\{ (v, \Xi) \in L^2 \times L^p \times L^2 \times L^2 : \text{ such that } \forall \varphi \in C^\infty(\mathbb{T}^3) \\ \int_{\mathbb{T}^3} \left(\frac{\Xi_A - \Xi_A^{j-1}}{h} \right) \varphi dx = - \int_{\mathbb{T}^3} \left(\frac{\partial \Psi^A}{\partial F_{i\alpha}}(F^{j-1}) v_i \right) \partial_\alpha \varphi dx \Big\}\end{aligned}$$

The minimizer satisfies the Euler-Lagrange equation (2.30)₁ in the sense of distributions, i.e.,

$$\int \varphi \frac{1}{h} (v_i - v_i^{j-1}) dx = - \int g_{i\alpha}(\Xi, F^{j-1}) \partial_\alpha \varphi dx$$

for all smooth φ . Furthermore the constraints

$$\partial_{\alpha i} Z = 0$$

$$\partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0$$

are preserved by the map

$$S_h : (v^{j-1}, F^{j-1}, Z^{j-1}, w^{j-1}) \rightarrow (v, F, Z, w),$$

the solution operator induced by the lemma. In fact if F^{j-1} is a gradient then so is F , and thus we can assert the existence of a $W^{1,p}$ function $y : \mathbb{T}^3 \rightarrow \mathbb{R}^3$ such that $\partial_\alpha y_i = F_{i\alpha}$.

In addition the iterates satisfy the following uniform estimate:

Lemma 2.2 ([15], p. 335). *Let $\Theta^{j-1} = (v^{j-1}, F^{j-1}, Z^{j-1}, w^{j-1})$ and $\Theta = (v, F, Z, w)$ be as in Lemma 2.1. If G is strictly convex function, i.e., if $\exists \gamma > 0$ such that $\nabla^2 G \geq \gamma$, then there exists $c > 0$ such that*

$$\int_{\Omega} \left(\eta(\Theta) + c|\Theta - \Theta^{j-1}|^2 \right) dx \leq \int_{\Omega} \eta(\Theta^{j-1}) dx.$$

Corollary 1 ([15], p. 335). *The iterates $\Theta^j = (v^j, F^j, Z^j, w^j)$, satisfy the energy dissipation inequality*

$$\frac{1}{h} \left(\eta(\Theta^j) - \eta(\Theta^{j-1}) \right) - \partial_\alpha (g_{i\alpha}(\Xi^j, F^{j-1}) v_i^j) \leq 0 \quad \text{for } j \geq 1$$

in the sense of distributions. There exists a number E_0 , determined by the initial data such that

$$\sup_{j \geq 0} \left(\|v^j\|_{L^2_{dx}}^2 + \int_{\mathbb{T}^3} G(\Xi^j) dx \right) + \sum_{j=1}^{\infty} \|\Theta^j - \Theta^{j-1}\|_{L^2_{dx}}^2 \leq E_0. \quad (2.31)$$

Let $(v^j, \Xi^j) = (v^j, F^j, Z^j, w^j)$, defined on the torus \mathbb{T}^3 , be the iterates constructed from the minimization process, $j = 0, 1, 2, \dots$. The iterates F^j are gradients, so we construct functions $y^j : \mathbb{T}^3 \rightarrow \mathbb{R}^3$ such that $\partial_\alpha y_i^j = F_{i\alpha}^j$. By selecting the integration constants appropriately (and choosing y^{-1} by extrapolation), the iterates y^j satisfy the identities

$$\frac{1}{h} (y^j - y^{j-1}) = v^j.$$

Following [15] construct the time-continuous, piecewise linear interpolates $V^{(h)}$, $\Xi^{(h)}$ given by (suppressing the explicit dependence on x of the iterates)

$$\begin{aligned} V^{(h)}(t) &= \sum_{j=1}^{\infty} \mathcal{X}^j(t) \left(v^{j-1} + \frac{t - h(j-1)}{h} (v^j - v^{j-1}) \right) \\ \Xi^{(h)}(t) &= (F^{(h)}, Z^{(h)}, w^{(h)})(t) \\ &= \sum_{j=1}^{\infty} \mathcal{X}^j(t) \left(\Xi^{j-1} + \frac{t - h(j-1)}{h} (\Xi^j - \Xi^{j-1}) \right), \end{aligned} \quad (2.32)$$

and the piecewise constant interpolates $v^{(h)}$, $\xi^{(h)}$ by

$$\begin{aligned} v^{(h)}(t) &= \sum_{j=1}^{\infty} \mathcal{X}^j(t) v^j \\ \xi^{(h)}(t) &= (f^{(h)}, z^{(h)}, \omega^{(h)})(t) = \sum_{j=1}^{\infty} \mathcal{X}^j(t) \Xi^j \end{aligned} \quad (2.33)$$

where $\mathcal{X}^j(t)$ is the characteristic function of the interval $I_j := [(j-1)h, jh)$. Finally, construct the piecewise linear approximation of the motion

$$Y^h(t) = \sum_{j=1}^{\infty} \mathcal{X}^j(t) \left(y^{j-1} + \frac{t - h(j-1)}{h} (y^j - y^{j-1}) \right) \quad (2.34)$$

and note the identities

$$\partial_t Y_i^h = v_i^h, \quad \partial_\alpha Y_i^h = F_{i\alpha}^h.$$

The approximate solutions (2.32), (2.33) and (2.34) give rise to measure valued solutions of systems (2.28) and (2.1) respectively as $h \rightarrow 0$. We first state the preliminary result on (weak) convergence of the approximates.

Lemma 2.3 ([15], p. 337). *The approximate solutions*

$$(V^h, F^h, Z^h, W^h) \quad \text{and} \quad (v^h, f^h, z^h, w^h)$$

are uniformly bounded in $L^\infty(L^2) \oplus L^\infty(L^p) \oplus L^\infty(L^q) \oplus L^\infty(L^r)$. Thus, there exists a subsequence in h and limit points $y : Q_\infty \rightarrow \mathbb{R}^3$ and $(v, \Xi) : Q_\infty \rightarrow \mathbb{R}^{22}$ with

$$y \in W^{1,\infty}(L^2) \cap L^\infty(W^{1,p}),$$

$$(v, \Xi) = (v, F, Z, w) \in L^\infty(L^2) \oplus L^\infty(L^p) \oplus L^\infty(L^q) \oplus L^\infty(L^r)$$

for all $T > 0$, and such that along the said subsequence

$$Y^h \rightarrow y \quad \text{strongly in } L^2_{loc}(\mathbb{T}^3) \text{ and a.e.}$$

$$(V^h, v^h, \Xi^h, \xi^h) \rightharpoonup (v, v, \Xi, \Xi)$$

weak \star in $L^\infty_{loc}(\mathbb{R}; [L^2]^2 \oplus [L^p \oplus L^q \oplus L^r]^2(\mathbb{T}^3))$, and

$$v_i = \partial_t y_i, \quad F_{i\alpha} = \partial_\alpha y_i.$$

In conclusion, we state the main result of [15].

Theorem 2.1 ([15], p. 332). *The discretization (2.30) can be solved for all $h > 0$ by a constrained minimization method, and has the property that the energy*

$$\mathcal{E}^j = \int_{\Omega} \left(\frac{1}{2} |v^j|^2 + G(\Xi^j) \right) dx.$$

is decreasing in j . As $h \rightarrow 0$ the approximations generate a measure-valued solution to (2.28) for which the momentum equation (2.28)₁ is satisfied in a measure-valued sense, but the constraint equation (2.28)₂ is satisfied in the classical weak sense. To be precise, there exists

$$(v, \Xi) = (v, F, Z, w) \in L^\infty(L^2) \oplus L^\infty(L^p) \oplus L^\infty(L^q) \oplus L^\infty(L^r)$$

and a Young measure $(\nu_{x,t})_{x,t \in Q_\infty}$ such that for $i = 1, \dots, 3$

$$- \int \phi(0, x) v_i(0, x) dx + \int v_i \partial_t \phi dx dt = \int \langle \nu, g_{i\alpha} \rangle \partial_\alpha \phi dx dt \quad (2.35)$$

and for $A = 1, \dots, 19$

$$- \int \phi(0, x) \Xi_A(0, x) dx + \int \Xi_A \partial_t \phi dx dt = \int \left(\frac{\partial \Psi^A}{\partial F_{i\alpha}}(F) v_i \right) \partial_\alpha \phi dx dt \quad (2.36)$$

for all smooth ϕ , compactly supported in time. Furthermore, there exists a map y , with space and time derivatives F, v respectively, such that (2.1) is satisfied in the measure-valued sense.

Remark 2.1. The spatially periodic map $y(x, t) : \mathbb{T}^3 \times [0, \infty) \rightarrow \mathbb{R}^3$ introduced in Lemma 2.3 cannot be interpreted as a motion since its invertibility fails. However, one should view the scheme as a tool which approximates the solution of elastodynamics $y : \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}^3$ whose gradient $F = \nabla y$ and velocity $v = \partial_t y$ are periodic in space. Given spatially periodic v and F the map y is obtained via integration and, in general, is not periodic (in space). For instance, one can write the motion as $y(x, t) = x + u(x, t)$ with u denoting the displacement field and search for y for which $u(\cdot, t)$ is periodic. In this case, $y(\cdot, t)$ itself is not a periodic map but its velocity and gradient are.

Remark 2.2. One of the shortcomings of the scheme is the fact that it generates only a measure-valued solution. Nevertheless, the approximating scheme (2.42) has regular weak solutions that decrease the energy. Also in cases with better compactness properties, such as the equations of nonlinear viscoelasticity, the method of time-discretization produces classical weak solutions [13].

Another shortcoming is that it is not required that $\det F > 0$, and therefore strictly speaking the map $y(x, t)$, reconstructed from the periodic (v, F) , may not be interpreted as an elastic motion.

Both of these deficiencies can be overcome in the one-dimensional case: The application of the method of compensated compactness [30] to the one-dimensional analogue of the present approximation scheme yields regular weak solutions that dissipate all convex entropies [14].

2.2 Convergence of the Variational Scheme

The objective of the present work is to show that the approximation scheme of [15] converges to the classical solution of the elastodynamics system before the formation of shocks; see [23]. In particular, we consider a smooth solution $\bar{\Theta} = (\bar{V}, \bar{\Xi})$ of the extended elasticity system (2.28) defined on $[0, T] \times \mathbb{T}^3$ and show that the approximate solution $\Theta^{(h)}$ constructed via the iterates (v^j, Ξ^j) of (2.30) converges to $\bar{\Theta} = (\bar{V}, \bar{\Xi})$ at a convergence rate $O(h)$.

2.2.1 Assumptions and Notations.

Due to technical difficulties, we narrow the class of stored energies functions used in [15] by modifying the hypothesis (H1*) – (H4*).

Assumptions. As in [15], the spatial domain Ω is taken to be the three-dimensional torus \mathbb{T}^3 . The indices i, α, \dots generally run over $1, \dots, 3$ while A, B, \dots run over $1, \dots, 19$. Finally, we impose the following convexity and growth assumptions on G :

(H1) $G \in C^3(M^{3 \times 3} \times M^{3 \times 3} \times \mathbb{R}; [0, \infty))$ is of the form

$$G(\Xi) = H(F) + R(\Xi) \tag{2.37}$$

with $H \in C^3(M^{3 \times 3}; [0, \infty))$ and $R \in C^3(M^{3 \times 3} \times M^{3 \times 3} \times \mathbb{R}; [0, \infty))$ strictly convex satisfying

$$\kappa|F|^{p-2}|z|^2 \leq z^T \nabla^2 H(F) z \leq \kappa'|F|^{p-2}|z|^2, \quad \forall z \in \mathbb{R}^9$$

and $\gamma I \leq \nabla^2 R \leq \gamma' I$ for some fixed $\gamma, \gamma', \kappa, \kappa' > 0$ and $p \in (6, \infty)$.

$$(H2) \quad G(\Xi) \geq c_1|F|^p + c_2|Z|^2 + c_3|w|^2 - c_4.$$

$$(H3) \quad G(\Xi) \leq c_5(|F|^p + |Z|^2 + |w|^2 + 1).$$

$$(H4) \quad |G_F|^{\frac{p}{p-1}} + |G_Z|^{\frac{p}{p-2}} + |G_w|^{\frac{p}{p-3}} \leq c_6(|F|^p + |Z|^2 + |w|^2 + 1).$$

$$(H5) \quad \left| \frac{\partial^3 H}{\partial F_{i\alpha} \partial F_{ml} \partial F_{rs}} \right| \leq c_7|F|^{p-3} \quad \text{and} \quad \left| \frac{\partial^3 R}{\partial \Xi_A \partial \Xi_B \partial \Xi_D} \right| \leq c_8.$$

Remark 2.3. The essential difference between hypothesis (H1) – (H5) from (H1*) – (H5*) is that the function G is split into the sum of two functions, see (H1), one of which is quadratic in Ξ and the other is a function of F whose growth is of order $p \in (6, \infty)$. This imposes the condition $r = q = 2$. The reason for doing so is the technical one: The solutions $(\bar{v}, \bar{\Xi})$ to the equations (2.28) have the property that if the constraint $\bar{\Xi} = \Psi(\bar{F})$ holds at $t = 0$ then it holds for all times. By contrast, the approximates $(V^{(h)}, \Xi^{(h)})$ do not have this property anymore. This presents various new technical difficulties which we were not able overcome without narrowing the class of polyconvex stored energies.

Notations. To simplify notation we denote

$$L^p = L^p(\mathbb{T}^3) \quad \text{and} \quad W^{1,p} = W^{1,p}(\mathbb{T}^3).$$

We also write

$$\begin{aligned} G_{,A}(\Xi) &= \frac{\partial G}{\partial \Xi_A}(\Xi), & R_{,A}(\Xi) &= \frac{\partial R}{\partial \Xi_A}(\Xi), \\ H_{,i\alpha}(F) &= \frac{\partial H}{\partial F_{i\alpha}}(F), & \Psi_{,i\alpha}^A(F) &= \frac{\partial \Psi^A}{\partial F_{i\alpha}}(F). \end{aligned}$$

In addition, for each $i, \alpha = 1, 2, 3$ we set

$$g_{i\alpha}(\Xi, F^*) = \frac{\partial G}{\partial \Xi_A}(\Xi) \frac{\partial \Psi^A}{\partial F_{i\alpha}}(F^*), \quad F^* \in \mathbb{R}^9, \Xi \in \mathbb{R}^{19} \quad (2.38)$$

(where we use the summation convention over repeated indices) and denote the corresponding fields $g_i : \mathbb{R}^{19} \times \mathbb{R}^9 \rightarrow \mathbb{R}^3$ by

$$g_i(\Xi, F^*) := (g_{i1}, g_{i2}, g_{i3})(\Xi, F^*). \quad (2.39)$$

Properties of the iterates. Since hypotheses (H1) – (H5) are the restriction of (H1*) – (H4*), we are able to apply the variational method proposed in [15] in the case of polyconvex stored energy $W(F) = G(\Psi(F))$ with G satisfying (H1) – (H4). Thus, given initial data

$$\Theta^0 := (v^0, \Xi^0) = (v^0, F^0, Z^0, w^0) \in L^2 \times L^p \times L^2 \times L^2 \quad (2.40)$$

and fixed $h > 0$, the variational method proposed in [15] provides the sequence of successive iterates

$$\Theta^j := (v^j, \Xi^j) = (v^j, F^j, Z^j, w^j) \in L^2 \times L^p \times L^2 \times L^2, \quad j \geq 1 \quad (2.41)$$

with the following properties (see Lemmas 2.1, 2.2 and Corollary 1):

(P 1) The iterate (v^j, Ξ^j) is the unique minimizer of the functional

$$\mathcal{J}(v, \Xi) = \int_{\mathbb{T}^3} \left(\frac{1}{2} |v - v^{j-1}|^2 + G(\Xi) \right) dx$$

over the weakly closed affine subspace

$$\mathcal{C} = \left\{ (v, \Xi) \in L^2 \times L^p \times L^2 \times L^2 : \text{ such that } \forall \varphi \in C^\infty(\mathbb{T}^3) \right. \\ \left. \int_{\mathbb{T}^3} \left(\frac{\Xi_A - \Xi_A^{j-1}}{h} \right) \varphi dx = - \int_{\mathbb{T}^3} \left(\frac{\partial \Psi^A}{\partial F_{i\alpha}} (F^{j-1}) v_i \right) \partial_\alpha \varphi dx \right\}.$$

(P 2) For each $j \geq 1$ the iterates satisfy

$$\begin{aligned} \frac{v_i^j - v_i^{j-1}}{h} &= \partial_\alpha \left(\frac{\partial G}{\partial \Xi_A}(\Xi^j) \frac{\partial \Psi^A}{\partial F_{i\alpha}}(F^{j-1}) \right) \\ \frac{\Xi_A^j - \Xi_A^{j-1}}{h} &= \partial_\alpha \left(\frac{\partial \Psi^A}{\partial F_{i\alpha}}(F^{j-1}) v_i^j \right) \end{aligned} \quad \text{in } \mathcal{D}'(\mathbb{T}^3). \quad (2.42)$$

(P 3) If F^0 is a gradient, then so is $F^j \ \forall j \geq 1$.

(P 4) Iterates $v^j, j \geq 1$ have higher regularity: $v^j \in W^{1,p}(\mathbb{T}^3), \forall j \geq 1$.

(P 5) There exists $E_0 > 0$ determined by the initial data such that

$$\sup_{j \geq 0} \left(\|v^j\|_{L_{dx}^2}^2 + \int_{\mathbb{T}^3} G(\Xi^j) dx \right) + \sum_{j=1}^{\infty} \|\Theta^j - \Theta^{j-1}\|_{L_{dx}^2}^2 \leq E_0. \quad (2.43)$$

Time-continuous iterates. Let $(v^j, \Xi^j), j \geq 0$ satisfy (2.40), (2.41). Define the time-continuous, piecewise linear interpolates $\Theta^{(h)} := (V^{(h)}, \Xi^{(h)})$ by

$$\begin{aligned} V^{(h)}(t) &= \sum_{j=1}^{\infty} \mathcal{X}^j(t) \left(v^{j-1} + \frac{t - h(j-1)}{h} (v^j - v^{j-1}) \right) \\ \Xi^{(h)}(t) &= (F^{(h)}, Z^{(h)}, w^{(h)})(t) \\ &= \sum_{j=1}^{\infty} \mathcal{X}^j(t) \left(\Xi^{j-1} + \frac{t - h(j-1)}{h} (\Xi^j - \Xi^{j-1}) \right), \end{aligned} \quad (2.44)$$

and the piecewise constant interpolates $\theta^{(h)} := (v^{(h)}, \xi^{(h)})$ and $\tilde{f}^{(h)}$ by

$$\begin{aligned} v^{(h)}(t) &= \sum_{j=1}^{\infty} \mathcal{X}^j(t) v^j \\ \xi^{(h)}(t) &= (f^{(h)}, z^{(h)}, \omega^{(h)})(t) = \sum_{j=1}^{\infty} \mathcal{X}^j(t) \Xi^j \\ \tilde{f}^{(h)}(t) &= \sum_{j=1}^{\infty} \mathcal{X}^j(t) F^{j-1} \end{aligned} \quad (2.45)$$

where $\mathcal{X}^j(t)$ is the characteristic function of the interval $I_j := [(j-1)h, jh)$.

Remark 2.4. Notice that $\tilde{f}^{(h)}$ is the time-shifted version of $f^{(h)}$. It is used later in defining a relative entropy flux, as well as the time-continuous equations (2.56).

2.2.2 Statement of the Main Results

In this section we state the main result on convergence. It asserts that the interpolates $\Theta^{(h)} = (V^{(h)}, \Xi^{(h)})$ obtained via the variational scheme converge to the solution of extended polyconvex elastodynamics (2.28) as long as the limit solution $\bar{\Theta} = (\bar{V}, \bar{\Xi})$ remains smooth.

Relative entropy method. For the proof of convergence we employ the relative entropy method developed for convex entropies in [9, 16] and adapted for the system of polyconvex elasticity in [21] using the embedding to the system (2.28). In this work, the difference between $\Theta^{(h)}$ and $\bar{\Theta}$ is controlled by monitoring the evolution of the relative entropy

$$\begin{aligned} \eta^r(\Theta^{(h)}, \bar{\Theta}) &:= \eta(\Theta^{(h)}) - \eta(\bar{\Theta}) - \nabla \eta(\bar{\Theta})(\Theta^{(h)} - \bar{\Theta}) \\ &= \frac{1}{2}|V^{(h)} - \bar{V}|^2 + G(\Xi^{(h)}) - G(\bar{\Xi}) - \nabla G(\bar{\Xi})(\Xi^{(h)} - \bar{\Xi}) \end{aligned} \quad (2.46)$$

for which the associated relative flux will turn out to be

$$q_\alpha^r(\theta^{(h)}, \tilde{f}^{(h)}, \bar{\Theta}) := (v_i^{(h)} - \bar{V}_i)(G_{,A}(\xi^{(h)}) - G_{,A}(\bar{\Xi})) \Psi_{,i\alpha}^A(\tilde{f}^{(h)}). \quad (2.47)$$

First we prove that the entropy pair η^r, q^r satisfies the identity (2.51) in the sense of distributions. Then, we establish control of the function

$$\mathcal{E}(t) := \int_{\Omega} \left((1 + |F^{(h)}|^{p-2} + |\bar{F}|^{p-2}) |F^{(h)} - \bar{F}|^2 + |\Theta^{(h)} - \bar{\Theta}|^2 \right) dx,$$

equivalent to the relative entropy η^r , and prove the estimate

$$\mathcal{E}(\tau) \leq C(\mathcal{E}(0) + h), \quad \tau \in [0, T],$$

which provides the result.

Main Convergence Theorem. *Let W be defined by (2.17) with G satisfying (H1) – (H5). Let $\Theta^{(h)} = (V^{(h)}, \Xi^{(h)})$, $\theta^{(h)} = (v^{(h)}, \xi^{(h)})$ and $\tilde{f}^{(h)}$ be the interpolates defined via (2.44), (2.45) and induced by the sequence of spatial iterates*

$$\Theta^j = (v^j, \Xi^j) = (v^j, F^j, Z^j, w^j) \in L^2 \times L^p \times L^2 \times L^2, \quad j \geq 0 \quad (2.48)$$

which satisfy (P1)-(P5). Let $\bar{\Theta} = (\bar{V}, \bar{\Xi}) = (\bar{V}, \bar{F}, \bar{Z}, \bar{w})$ be a smooth solution of (2.28) defined on $\mathbb{T}^3 \times [0, T]$ and emanate from the data $\bar{\Theta}^0 = (\bar{V}^0, \bar{F}^0, \bar{Z}^0, \bar{w}^0)$.

Assume also that F^0, \bar{F}^0 are gradients. Then:

- (a) *The relative entropy $\eta^r = \eta^r(\Theta^{(h)}, \bar{\Theta})$ satisfies (2.51). Furthermore, there exist constants $\mu, \mu' > 0$ such that*

$$\mu \mathcal{E}(t) \leq \int_{\mathbb{T}^3} \eta^r(x, t) dx \leq \mu' \mathcal{E}(t), \quad t \in [0, T]$$

where

$$\mathcal{E}(t) := \int_{\mathbb{T}^3} \left((1 + |F^{(h)}|^{p-2} + |\bar{F}|^{p-2}) |F^{(h)} - \bar{F}|^2 + |\Theta^{(h)} - \bar{\Theta}|^2 \right) dx.$$

- (b) *There exists $\varepsilon > 0$ and $C = C(T, \bar{\Theta}, E_0, \mu, \mu', \varepsilon) > 0$ such that $\forall h \in (0, \varepsilon)$*

$$\mathcal{E}(\tau) \leq C(\mathcal{E}(0) + h), \quad \tau \in [0, T].$$

Moreover, if the data satisfy $\mathcal{E}^{(h)}(0) \rightarrow 0$ as $h \downarrow 0$, then

$$\sup_{t \in [0, T]} \int_{\mathbb{T}^3} (|\Theta^{(h)} - \bar{\Theta}|^2 + |F^{(h)} - \bar{F}|^2 (1 + |F^{(h)}|^{p-2} + |\bar{F}|^{p-2})) dx \rightarrow 0$$

as $h \downarrow 0$.

Corollary. *Let $\Theta^{(h)} = (V^{(h)}, \Xi^{(h)})$ be as in the main theorem. Let (\bar{V}, \bar{F}) be a smooth solution of (2.22) with $\bar{F}(\cdot, 0)$ a gradient and $\bar{\Theta} = (\bar{V}, \Psi(\bar{F}))$. Assume that initial data satisfy $\Theta^{(h)}(\cdot, 0) = \bar{\Theta}(\cdot, 0)$. Then*

$$\sup_{t \in [0, T]} \left(\|V - \bar{V}\|_{L^2(\mathbb{T}^3)}^2 + \|\Xi^{(h)} - \Psi(\bar{F})\|_{L^2(\mathbb{T}^3)}^2 + \|F^{(h)} - \bar{F}\|_{L^p(\mathbb{T}^3)}^p \right) = O(h).$$

Remark 2.5. The smooth solution $\bar{\Theta} = (\bar{V}, \bar{\Xi})$ to the extended system (2.28) is provided beforehand. A natural question arises whether such a solution exists. We briefly discuss the existence theory for (2.2) on the torus \mathbb{T}^3 . In [12] energy methods are used to establish local (in time) existence of smooth solutions to certain initial-boundary value problem that apply to the system of nonlinear elastodynamics (2.1) with rank-1 convex stored energy. More precisely, for a bounded domain $\Omega \subset \mathbb{R}^n$ with the smooth boundary $\partial\Omega$ the authors establish ([12, Theorem 5.2]) the existence of a unique displacement field $y(\cdot, t)$ satisfying (2.1) in $\Omega \times [0, T]$ together with boundary conditions $y(x, t) = 0$ on $\partial\Omega \times [0, T]$ and initial conditions $y(\cdot, 0) = y_0$ and $y_t(\cdot, 0) = y_1$ whenever $T > 0$ is small enough and the initial data lie in a compact set. One may get a counterpart of this result for solutions on \mathbb{T}^3 since the methods in [12] are developed in the abstract framework: a quasi-linear partial differential equation is viewed as an abstract differential equation with initial value problem set on an interpolated scale of separable Hilbert spaces $\{H_\gamma\}_{\gamma \in [0, m]}$ with $m \geq 2$. To be precise, the spaces satisfy $H_\gamma = [H_0, H_m]_{\gamma/m}$ and the desired solution $u(t)$ of an abstract differential equation is assumed to be taking values in $H_m \cap V$, where V , a closed subspace of H_1 , is designated to accommodate the boundary

conditions (*cf.* [12, Sec.2]). By choosing appropriate spaces, namely

$$H_\gamma = [L^2(\mathbb{T}^3), W^{m,2}(\mathbb{T}^3)]_{\gamma/m} \quad \text{and} \quad V = H_1 = W^{1,2}(\mathbb{T}^3),$$

and requiring strong ellipticity (*cf.* [12, Sec.5]) for the stored energy one may apply [12, Thm 4.1] to conclude the local existence of smooth solutions on the torus \mathbb{T}^3 to the system of elastodynamics (2.1) and hence to (2.2). Since strong polyconvexity implies strong ellipticity [3], the same conclusion holds for the case of polyconvex energy which is used here.

2.2.3 Relative Entropy Identity

The goal of this section is to derive an identity for a relative energy among the two solutions. For the rest of the chapter, we suppress the dependence on h to simplify notations and, *cf.* Main Theorem, assume:

- (1) $\Theta = (V, \Xi)$, $\theta = (v, \xi)$, \tilde{f} are the approximates defined by (2.44) and (2.45).
- (2) $\bar{\Theta} = (\bar{V}, \bar{\Xi}) = (\bar{V}, \bar{F}, \bar{Z}, \bar{w})$ is a smooth solution of (2.28) defined on $\mathbb{T}^3 \times [0, T]$ where $T > 0$ is finite.

We now state two elementary lemmas used in our further computations. The first one extends the null-Lagrangian properties while the second one provides the rule for the divergence of the product in the nonsmooth case.

Lemma 2.4 (Null-Lagrangian properties). *Assume $q > 2$ and $r \geq \frac{q}{q-2}$. Then,*

if $u \in W^{1,q}(\mathbb{T}^3; \mathbb{R}^3)$, $z \in W^{1,r}(\mathbb{T}^3)$, we have

$$\begin{aligned} \partial_\alpha \left(\frac{\partial \Psi^A}{\partial F_{i\alpha}} (\nabla u) \right) &= 0 \\ \partial_\alpha \left(\frac{\partial \Psi^A}{\partial F_{i\alpha}} (\nabla u) z \right) &= \frac{\partial \Psi^A}{\partial F_{i\alpha}} (\nabla u) \partial_\alpha z \end{aligned} \quad \text{in } \mathcal{D}'(\mathbb{T}^3) \quad (2.49)$$

for each $i = 1, \dots, 3$ and $A = 1, \dots, 19$.

Proof. Observe that

$$\Psi_{,i\alpha}(\nabla u) \leq 1 + |\nabla u| + |\nabla u|^2 \quad \Rightarrow \quad \frac{\partial \Psi^A}{\partial F_{i\alpha}}(\nabla u) \in L^{q/2}(\mathbb{T}^3).$$

Hence by (2.26) and the density argument we get (2.49)₁. Next, notice that

$$\frac{\partial \Psi^A}{\partial F_{i\alpha}}(\nabla u) z, \quad \frac{\partial \Psi^A}{\partial F_{i\alpha}}(\nabla u) \partial_\alpha z \in L^1(\mathbb{T}^3).$$

Then taking arbitrary $\varphi \in C^\infty(\mathbb{T}^3)$ we obtain

$$\begin{aligned} \int_{\mathbb{T}^3} \left(\frac{\partial \Psi^A}{\partial F_{i\alpha}}(\nabla u) z \right) \partial_\alpha \varphi \, dx \\ = \int_{\mathbb{T}^3} \left(\frac{\partial \Psi^A}{\partial F_{i\alpha}}(\nabla u) \right) \partial_\alpha (z \varphi) \, dx - \int_{\mathbb{T}^3} \left(\frac{\partial \Psi^A}{\partial F_{i\alpha}}(\nabla u) \partial_\alpha z \right) \varphi \, dx = I_1 - I_2. \end{aligned}$$

Since $z\varphi \in W_0^{1,r} \cap W^{1,q^*}$, the property (2.49)₁ and the density argument imply

$I_1 = 0$ and hence

$$\int_{\mathbb{T}^3} \left(\frac{\partial \Psi^A}{\partial F_{i\alpha}}(\nabla u) z \right) \partial_\alpha \varphi \, dx = -I_2 = \int_{\mathbb{T}^3} \left(\frac{\partial \Psi^A}{\partial F_{i\alpha}}(\nabla u) \partial_\alpha z \right) \varphi \, dx.$$

□

Lemma 2.5 (Product rule). *Let $q \in (1, \infty)$ and $q' = \frac{q}{q-1}$. Assume*

$$f \in W^{1,q}(\mathbb{T}^3), \quad h \in L^{q'}(\mathbb{T}^3; \mathbb{R}^3) \quad \text{and} \quad z = \operatorname{div} h \in L^{q'}(\mathbb{T}^3).$$

Then $fh \in L^1(\mathbb{T}^3; \mathbb{R}^3)$, $\operatorname{div}(fh) \in L^1(\mathbb{T}^3)$ and

$$\operatorname{div}(fh) = f \operatorname{div} h + \nabla f h \quad \text{in } \mathcal{D}'(\mathbb{T}^3). \quad (2.50)$$

Proof. First, observe that

$$h \in L^{q'}(\mathbb{T}^3; \mathbb{R}^3), f \in L^q(\mathbb{T}^3) \Rightarrow fh \in L^1(\mathbb{T}^3; \mathbb{R}^3).$$

Then, since $f \in W^{1,q}(\mathbb{T}^3)$, for each $\varphi \in C^\infty(\mathbb{T}^3)$ we obtain

$$-\int_{\mathbb{T}^3} fh_\alpha \partial_\alpha \varphi \, dx = -\int_{\mathbb{T}^3} h_\alpha \partial_\alpha (f\varphi) \, dx + \int_{\mathbb{T}^3} (h_\alpha \partial_\alpha f) \varphi \, dx.$$

Further, notice that $f\varphi \in W_0^{1,q}(\mathbb{T}^3)$ and hence

$$-\int_{\mathbb{T}^3} h_\alpha \partial_\alpha (f\varphi) \, dx = \int_{\mathbb{T}^3} z(f\varphi) \, dx$$

where we used the density argument and the assumption that $z = \operatorname{div} h \in L^{q'}(\mathbb{T}^3)$.

Hence

$$-\int_{\mathbb{T}^3} fh_\alpha \partial_\alpha \varphi \, dx = \int_{\mathbb{T}^3} (zf + h_\alpha \partial_\alpha f) \varphi \, dx$$

and this proves (2.50). Finally, notice that

$$zf, h_\alpha \partial_\alpha f \in L^1(\mathbb{T}^3) \Rightarrow \operatorname{div}(fh) \in L^1(\mathbb{T}^3).$$

and this finishes the proof. □

Lemma 2.6 (Relative entropy identity). *For almost all $t \in [0, T]$*

$$\partial_t \eta^r - \operatorname{div} q^r = Q - \frac{1}{h} \sum_{j=1}^{\infty} \mathcal{X}^j(t) D^j + S \quad \text{in } \mathcal{D}'(\mathbb{T}^3) \quad (2.51)$$

where

$$\begin{aligned} Q := & \partial_\alpha (G_{,A}(\bar{\Xi})) (\Psi_{,i\alpha}^A(F) - \Psi_{,i\alpha}^A(\bar{F})) (V_i - \bar{V}_i) \\ & + \partial_\alpha \bar{V}_i (G_{,A}(\Xi) - G_{,A}(\bar{\Xi})) (\Psi_{,i\alpha}^A(F) - \Psi_{,i\alpha}^A(\bar{F})) \\ & + \partial_\alpha \bar{V}_i (G_{,A}(\Xi) - G_{,A}(\bar{\Xi}) - G_{,AB}(\bar{\Xi})(\Xi - \bar{\Xi})_B) \Psi_{,i\alpha}^A(\bar{F}) \end{aligned} \quad (2.52)$$

estimates the difference between the two solutions,

$$D^j := (\nabla \eta(\theta) - \nabla \eta(\Theta)) \delta \Theta^j, \quad (2.53)$$

where $\delta \Theta^j := \Theta^j - \Theta^{j-1}$, are the dissipative terms, and

$$\begin{aligned} S := & \partial_\alpha (G_{,A}(\bar{\Xi})) \left[\Psi_{,i\alpha}^A(\bar{F})(v_i - V_i) \right. \\ & + (\Psi_{,i\alpha}^A(F) - \Psi_{,i\alpha}^A(\bar{F}))(v_i - V_i) \\ & + (\Psi_{,i\alpha}^A(\tilde{f}) - \Psi_{,i\alpha}^A(F))(v_i - V_i) \\ & \left. + (\Psi_{,i\alpha}^A(\tilde{f}) - \Psi_{,i\alpha}^A(F))(V_i - \bar{V}_i) \right] \\ & + \partial_\alpha \bar{V}_i \left[(G_{,A}(\xi) - G_{,A}(\Xi)) \Psi_{,i\alpha}^A(\bar{F}) \right. \\ & + (G_{,A}(\xi) - G_{,A}(\Xi)) (\Psi_{,i\alpha}^A(\tilde{f}) - \Psi_{,i\alpha}^A(F)) \\ & + (G_{,A}(\xi) - G_{,A}(\Xi)) (\Psi_{,i\alpha}^A(F) - \Psi_{,i\alpha}^A(\bar{F})) \\ & \left. + (G_{,A}(\Xi) - G_{,A}(\bar{\Xi})) (\Psi_{,i\alpha}^A(\tilde{f}) - \Psi_{,i\alpha}^A(F)) \right] \end{aligned} \quad (2.54)$$

is the error term.

Proof. Notice that by (2.44) for almost all $t \geq 0$

$$\begin{aligned} \partial_t V(\cdot, t) &= \sum_{j=1}^{\infty} \mathcal{X}^j(t) \frac{\delta v^j}{h}, \quad \delta v^j := v^j - v^{j-1} \\ \partial_t \Xi(\cdot, t) &= \sum_{j=1}^{\infty} \mathcal{X}^j(t) \frac{\delta \Xi^j}{h}, \quad \delta \Xi^j := \Xi^j - \Xi^{j-1}. \end{aligned} \quad (2.55)$$

Hence by (2.38), (2.42) and (2.55) we obtain for almost all $t \geq 0$

$$\begin{aligned} \partial_t V_i(\cdot, t) &= \operatorname{div}(g_i(\xi, \tilde{f})) \\ &\quad \text{in } \mathcal{D}'(\mathbb{T}^3). \\ \partial_t \Xi_A(\cdot, t) &= \partial_\alpha (\Psi_{,i\alpha}^A(\tilde{f}) v_i) \end{aligned} \quad (2.56)$$

Since $(\bar{V}, \bar{\Xi})$ is the smooth solution of (2.28), using (2.38) we also have

$$\begin{aligned} \partial_t \bar{V}_i &= \operatorname{div}(g_i(\bar{\Xi}, \bar{F})) \\ &\quad \text{in } \mathbb{T}^3 \times [0, T]. \\ \partial_t \bar{\Xi}_A &= \partial_\alpha (\Psi_{,i\alpha}^A(\bar{F}) \bar{V}_i) \end{aligned} \quad (2.57)$$

Further in the proof we will perform a series of calculations that hold for smooth functions. A technical difficulty arises, since the iterates (v^j, Ξ^j) , $j \geq 1$ satisfying (2.42) are, in general, not smooth. To bypass this we employ Lemmas 2.4 and 2.5 that provide the null-Lagrangian property and product rule in the smoothness class appropriate for the approximates $\Theta = (V, \Xi)$, $\theta = (v, \xi)$, \tilde{f} .

By assumption F^0 and \bar{F}^0 are gradients. Hence using (P3) we conclude that F^j , $j \geq 1$ are gradients. Furthermore, from (E1) it follows that \bar{F} remains a gradient $\forall t$. Thus, recalling (2.44)-(2.45), we have

$$F, f, \tilde{f} \text{ and } \bar{F} \text{ are gradients } \forall t \in [0, T]. \quad (2.58)$$

We also notice that by (2.18), (2.38), and (H4) we have for all $F^* \in \mathbb{R}^9$, $\Xi^\circ \in \mathbb{R}^{19}$

$$\begin{aligned} & |g_{i\alpha}(\Xi^\circ, F^*)|^{p'} \\ & \leq C_g \left(\left| \frac{\partial G}{\partial F_{i\alpha}} \right|^{\frac{p}{p-1}} + |F^*|^{\frac{p}{p-1}} \left| \frac{\partial G}{\partial Z_{k\gamma}} \right|^{\frac{p}{p-1}} + |F^*|^{\frac{2p}{p-1}} \left| \frac{\partial G}{\partial w} \right|^{\frac{p}{p-1}} \right) \\ & \leq C'_g \left(|F^*|^p + \left| \frac{\partial G}{\partial F_{i\alpha}} \right|^{\frac{p}{p-1}} + \left| \frac{\partial G}{\partial Z_{k\gamma}} \right|^{\frac{p}{p-2}} + \left| \frac{\partial G}{\partial w} \right|^{\frac{p}{p-3}} \right) \\ & \leq C''_g \left(|F^*|^p + |F^\circ|^p + |Z^\circ|^2 + |w^\circ|^2 + 1 \right) \end{aligned} \quad (2.59)$$

where $p \in (6, \infty)$ and $p' = \frac{p}{p-1}$. Hence (H2), (P4)-(P5), (2.45)₁ and Lemmas 2.4, 2.5 along with (2.56)₁ imply

$$\begin{aligned} \operatorname{div}(v_i g_i(\xi, \tilde{f})) &= v_i \partial_t V_i + \nabla v_i g_i(\xi, \tilde{f}) \\ \operatorname{div}(\bar{V}_i g_i(\xi, \tilde{f})) &= \bar{V}_i \partial_t V_i + \nabla \bar{V}_i g_i(\xi, \tilde{f}) \\ \operatorname{div}(v_i g_i(\bar{\Xi}, \tilde{f})) &= v_i \Psi_{,i\alpha}^A(\tilde{f}) \partial_\alpha (G_{,A}(\bar{\Xi})) + \nabla v_i g_i(\bar{\Xi}, \tilde{f}) \\ \operatorname{div}(\bar{V}_i g_i(\bar{\Xi}, \tilde{f})) &= \bar{V}_i \Psi_{,i\alpha}^A(\tilde{f}) \partial_\alpha (G_{,A}(\bar{\Xi})) + \nabla \bar{V}_i g_i(\bar{\Xi}, \tilde{f}). \end{aligned} \quad (2.60)$$

Similarly, by (P4), Lemma 2.4, (2.56)₂ and (2.58) we have the identity

$$\partial_t \Xi_A(t) = \Psi_{,i\alpha}^A(\tilde{f}) \partial_\alpha v_i. \quad (2.61)$$

Thus, using (2.29), (2.60)₁ and (2.61), we compute

$$\begin{aligned} \partial_t(\eta(\Theta)) &= V_i \partial_t V_i + G_{,A}(\Xi) \partial_t \Xi_A \\ &= (V_i - v_i) \partial_t V_i + (G_{,A}(\Xi) - G_{,A}(\xi)) \partial_t \Xi_A + \operatorname{div}(v_i g_i(\xi, \tilde{f})) \\ &= \frac{1}{h} \sum_{j=1}^{\infty} \mathcal{X}^j(t) (\nabla \eta(\Theta) - \nabla \eta(\theta)) \delta \Theta^j + \operatorname{div}(v_i g_i(\xi, \tilde{f})). \end{aligned}$$

Furthermore, by (2.60)₂ we have

$$\begin{aligned} \partial_t(\bar{V}_i(V_i - \bar{V}_i)) &= \partial_t \bar{V}_i(V_i - \bar{V}_i) + \bar{V}_i \partial_t V_i - \bar{V}_i \partial_t \bar{V}_i \\ &= \partial_t \bar{V}_i(V_i - \bar{V}_i) + \operatorname{div}(\bar{V}_i g_i(\xi, \tilde{f})) - \nabla \bar{V}_i g_i(\xi, \tilde{f}) - \frac{1}{2} \partial_t \bar{V}^2 \end{aligned}$$

while using (2.61) we obtain

$$\begin{aligned} \partial_t(G_{,A}(\bar{\Xi})(\Xi - \bar{\Xi})_A) &= \partial_t(G_{,A}(\bar{\Xi}))(\Xi - \bar{\Xi})_A + G_{,A}(\bar{\Xi}) \partial_t \Xi_A - \partial_t(G(\bar{\Xi})) \\ &= \partial_t(G_{,A}(\bar{\Xi}))(\Xi - \bar{\Xi})_A + \nabla v_i g_i(\bar{\Xi}, \tilde{f}) - \partial_t(G(\bar{\Xi})). \end{aligned}$$

Next, notice that by (2.38) and (2.47) we have

$$q^r = v_i g_i(\xi, \tilde{f}) - \bar{V}_i g_i(\xi, \tilde{f}) - v_i g_i(\bar{\Xi}, \tilde{f}) + \bar{V}_i g_i(\bar{\Xi}, \tilde{f}). \quad (2.62)$$

Hence by (2.29), (2.46), (2.53), (2.60) and the last four identities we obtain

$$\partial \eta^r - \operatorname{div} q^r = -\frac{1}{h} \sum_{j=1}^{\infty} \mathcal{X}^j(t) D^j + J \quad (2.63)$$

where

$$\begin{aligned} J &:= -\operatorname{div}(\bar{V}_i g_i(\bar{\Xi}, \tilde{f})) + \nabla \bar{V}_i g_i(\xi, \tilde{f}) \\ &\quad + \operatorname{div}(v_i g_i(\bar{\Xi}, \tilde{f})) - \nabla v_i g_i(\bar{\Xi}, \tilde{f}) \\ &\quad - \partial_t \bar{V}_i(V_i - \bar{V}_i) - \partial_t(G_{,A}(\bar{\Xi}))(\Xi - \bar{\Xi})_A. \end{aligned}$$

Consider now the term J . From (2.57)-(2.58) and Lemma 2.4 it follows that

$$\partial_t \bar{V}_i = \Psi_{,i\alpha}^A(\bar{F}) \partial_\alpha (G_{,A}(\bar{\Xi}))$$

$$\partial_t (G_{,A}(\bar{\Xi})) = G_{,AB}(\bar{\Xi}) \Psi_{,i\alpha}^B(\bar{F}) \partial_\alpha \bar{V}_i.$$

Then, (2.60)_{3,4} along with the last two identities and the fact that $G_{,AB} = G_{,BA}$ implies

$$\begin{aligned} J &= \partial_\alpha \bar{V}_i \left(g_{i\alpha}(\xi, \tilde{f}) - g_{i\alpha}(\bar{\Xi}, \tilde{f}) \right) \\ &\quad + \partial_\alpha (G_{,A}(\bar{\Xi})) \left(\Psi_{,i\alpha}^A(\tilde{f})(v_i - \bar{V}_i) - \Psi_{,i\alpha}^A(\bar{F})(V_i - \bar{V}_i) \right) \\ &\quad - G_{,AB}(\bar{\Xi})(\Xi - \bar{\Xi})_A \Psi_{,i\alpha}^B(\bar{F}) \partial_\alpha \bar{V}_i \\ &= \partial_\alpha \bar{V}_i \left(g_{i\alpha}(\xi, \tilde{f}) - g_{i\alpha}(\bar{\Xi}, \tilde{f}) - g_{i\alpha}(\Xi, \bar{F}) + g_{i\alpha}(\bar{\Xi}, \bar{F}) \right) \\ &\quad + \partial_\alpha (G_{,A}(\bar{\Xi})) \left(\Psi_{,i\alpha}^A(\tilde{f})(v_i - \bar{V}_i) - \Psi_{,i\alpha}^A(\bar{F})(V_i - \bar{V}_i) \right) \\ &\quad + \partial_\alpha \bar{V}_i \left(G_{,A}(\Xi) - G_{,A}(\bar{\Xi}) - G_{,AB}(\bar{\Xi})(\Xi - \bar{\Xi})_B \right) \Psi_{,i\alpha}^A(\bar{F}) \\ &=: J_1 + J_2 + J_3. \end{aligned} \tag{2.64}$$

Using (2.38) we rearrange the term J_1 as follows:

$$\begin{aligned} J_1 &= \partial_\alpha \bar{V}_i \left[(G_{,A}(\xi) - G_{,A}(\bar{\Xi})) \Psi_{,i\alpha}^A(\tilde{f}) - (G_{,A}(\Xi) - G_{,A}(\bar{\Xi})) \Psi_{,i\alpha}^A(\bar{F}) \right] \\ &= \partial_\alpha \bar{V}_i \left[(G_{,A}(\xi) - G_{,A}(\Xi)) (\Psi_{,i\alpha}^A(\tilde{f}) - \Psi_{,i\alpha}^A(F)) \right. \\ &\quad + (G_{,A}(\xi) - G_{,A}(\Xi)) (\Psi_{,i\alpha}^A(F) - \Psi_{,i\alpha}^A(\bar{F})) \\ &\quad + (G_{,A}(\xi) - G_{,A}(\Xi)) \Psi_{,i\alpha}^A(\bar{F}) \\ &\quad + (G_{,A}(\Xi) - G_{,A}(\bar{\Xi})) (\Psi_{,i\alpha}^A(\tilde{f}) - \Psi_{,i\alpha}^A(F)) \\ &\quad \left. + (G_{,A}(\Xi) - G_{,A}(\bar{\Xi})) (\Psi_{,i\alpha}^A(F) - \Psi_{,i\alpha}^A(\bar{F})) \right]. \end{aligned} \tag{2.65}$$

We also modify the term J_2 writing it in the following way:

$$\begin{aligned}
J_2 &= \partial_\alpha(G_{,A}(\bar{\Xi})) \left[\Psi_{,i\alpha}^A(\tilde{f})(v_i - \bar{V}_i) - \Psi_{,i\alpha}^A(\bar{F})(V_i - \bar{V}_i) \right] \\
&= \partial_\alpha(G_{,A}(\bar{\Xi})) \left[(\Psi_{,i\alpha}^A(F) - \Psi_{,i\alpha}^A(\bar{F}))(V_i - \bar{V}_i) \right. \\
&\quad + (\Psi_{,i\alpha}^A(\tilde{f}) - \Psi_{,i\alpha}^A(F))(V_i - \bar{V}_i) \\
&\quad + (\Psi_{,i\alpha}^A(\tilde{f}) - \Psi_{,i\alpha}^A(F))(v_i - V_i) \\
&\quad + (\Psi_{,i\alpha}^A(F) - \Psi_{,i\alpha}^A(\bar{F}))(v_i - V_i) \\
&\quad \left. + \Psi_{,i\alpha}^A(\bar{F})(v_i - V_i) \right].
\end{aligned} \tag{2.66}$$

By (2.64)-(2.66) we have $J = J_1 + J_2 + J_3 = Q + S$. Hence by (2.63) we get (2.51). \square

2.2.4 Proof of Main Convergence Theorem

The identity (2.51) is central to our paper. In this section, we estimate each of its terms and complete the proof via Gronwall's inequality.

Definition. Let $\Theta_1 = (V_1, \Xi_1), \Theta_2 = (V_2, \Xi_2) \in \mathbb{R}^{22}$. We set

$$d(\Theta_1, \Theta_2) = (1 + |F_1|^{p-2} + |F_2|^{p-2}) |F_1 - F_2|^2 + |\Theta_1 - \Theta_2|^2 \tag{2.67}$$

where $(F_1, Z_1, w_1) = \Xi_1, (F_2, Z_2, w_2) = \Xi_2 \in \mathbb{R}^{19}$.

Our first objective is to show that the relative entropy η^r can be equivalently represented by the function $d(\cdot, \cdot)$. Before we establish this relation, we prove an elementary lemma used in our further calculations:

Lemma 2.7. *Assume $q \geq 1$. Then for all $u, v \in \mathbb{R}^n$ and $\bar{\beta} \in [0, 1]$*

$$\int_0^{\bar{\beta}} \int_0^1 (1 - \beta) |u + \alpha(1 - \beta)(v - u)|^q d\alpha d\beta \geq c' \bar{\beta} (|u|^q + |v|^q) \tag{2.68}$$

with constant $c' > 0$ depending only on q and n .

Proof. Observe first that

$$\int_0^1 |u + \alpha(v - u)| d\alpha \geq \bar{c}(|u| + |v|), \quad \forall u, v \in \mathbb{R}^n \quad (2.69)$$

with $\bar{c} = \frac{1}{4\sqrt{n}}$. Then, applying Jensen's inequality and using (2.69), we get

$$\begin{aligned} & \int_0^{\bar{\beta}} \int_0^1 (1 - \beta) |u + \alpha(1 - \beta)(v - u)|^q d\alpha d\beta \\ & \geq \int_0^{\bar{\beta}} (1 - \beta) \left(\int_0^1 |u + \alpha((1 - \beta)v + \beta u - u)| d\alpha \right)^q d\beta \\ & \geq \bar{c}^q \int_0^{\bar{\beta}} (1 - \beta) (|u| + |(1 - \beta)v + \beta u|)^q d\beta \\ & \geq \frac{\bar{c}^q}{2} (|u|^q + |v|^q) \int_0^{\bar{\beta}} (1 - \beta)^{q+1} d\beta. \end{aligned}$$

Since $q \geq 1$ and $(1 - \bar{\beta}) \in [0, 1]$, we have

$$\int_0^{\bar{\beta}} (1 - \beta)^{q+1} d\beta = \frac{1 - (1 - \bar{\beta})^{q+2}}{q + 2} \geq \frac{\bar{\beta}}{q + 2}.$$

Combining the last two inequalities we obtain (2.68). \square

Lemma 2.8 (η^r -equivalence). *There exist constants $\mu, \mu' > 0$ such that*

$$\mu d(\Theta_1, \Theta_2) \leq \eta^r(\Theta_1, \Theta_2) \leq \mu' d(\Theta_1, \Theta_2) \quad (2.70)$$

for every $\Theta_1 = (V_1, \Xi_1), \Theta_2 = (V_2, \Xi_2) \in \mathbb{R}^{22}$.

Proof. Notice that

$$\begin{aligned} \eta^r(\Theta_1, \Theta_2) &= \eta(\Theta_1) - \eta(\Theta_2) - \nabla \eta(\Theta_2)(\Theta_1 - \Theta_2) \\ &= \int_0^1 \int_0^1 s(\Theta_1 - \Theta_2)^T (\nabla^2 \eta(\hat{\Theta})) (\Theta_1 - \Theta_2) ds d\tau. \end{aligned} \quad (2.71)$$

where

$$\hat{\Theta} = (\hat{V}, \hat{\Xi}) = (\hat{V}, \hat{F}, \hat{Z}, \hat{w}) := \Theta_2 + \tau s(\Theta_1 - \Theta_2), \quad \tau, s \in [0, 1].$$

Observe next that

$$\nabla_{\Xi} G = \begin{bmatrix} \nabla_F H & 0 & 0 \end{bmatrix} + \nabla_{\Xi} R \quad (2.72)$$

and therefore by (2.29)

$$\begin{aligned} & (\Theta_1 - \Theta_2)^T \nabla^2 \eta(\hat{\Theta})(\Theta_1 - \Theta_2) \\ &= |V_1 - V_2|^2 + (\Xi_1 - \Xi_2)^T \nabla^2 R(\hat{\Xi})(\Xi_1 - \Xi_2) \\ & \quad + (F_1 - F_2)^T \nabla^2 H(\hat{F})(F_1 - F_2). \end{aligned} \quad (2.73)$$

Then (H1), (2.71) and (2.73) imply

$$\begin{aligned} & \frac{1}{2} |V_1 - V_2|^2 + \frac{\gamma}{2} |\Xi_1 - \Xi_2|^2 + \kappa |F_1 - F_2|^2 \int_0^1 \int_0^1 s |\hat{F}|^{p-2} ds d\tau \\ & \leq \eta^r(\Theta_1, \Theta_2) \leq \\ & \frac{1}{2} |V_1 - V_2|^2 + \frac{\gamma'}{2} |\Xi_1 - \Xi_2|^2 + \kappa' |F_1 - F_2|^2 \int_0^1 \int_0^1 s |\hat{F}|^{p-2} ds d\tau. \end{aligned} \quad (2.74)$$

We now consider the integral term in (2.74). Recall that $\hat{F} = F_2 + \tau s(F_1 - F_2)$.

Then, estimating from above, we get

$$\int_0^1 \int_0^1 s |\hat{F}|^{p-2} ds d\tau \leq 2^{p-3} (|F_1|^{p-2} + |F_2|^{p-2})$$

while for the estimate from below we use Lemma 2.7 (with $s = 1 - \beta$ and $\bar{\beta} = 1$)

and obtain

$$\int_0^1 \int_0^1 s |\hat{F}|^{p-2} ds d\tau \geq c' (|F_1|^{p-2} + |F_2|^{p-2}).$$

Combining (2.74) with the two last inequalities we obtain (2.70). \square

Observe that the smoothness of $\bar{\Theta}$ implies that $\exists M = M(T) > 0$ such that

$$M \geq |\bar{\Theta}| + |\nabla_x \bar{\Theta}| + |\partial_t \bar{\Theta}|, \quad (x, t) \in \mathbb{T}^3 \times [0, T]. \quad (2.75)$$

Lemma 2.9 (\mathcal{E} -equivalence). *The relative entropy $\eta^r(\Theta, \bar{\Theta})$ and function $d(\Theta, \bar{\Theta})$ satisfy*

$$\eta^r(\Theta, \bar{\Theta}), d(\Theta, \bar{\Theta}) \in L^\infty([0, T]; L^1(\mathbb{T}^3)).$$

Moreover,

$$\mu \mathcal{E}(t) \leq \int_{\mathbb{T}^3} \eta^r(\Theta(x, t), \bar{\Theta}(x, t)) dx \leq \mu' \mathcal{E}(t), \quad \forall t \in [0, T]$$

where

$$\mathcal{E}(t) := \int_{\mathbb{T}^3} d(\Theta(x, t), \bar{\Theta}(x, t)) dx.$$

and constants μ, μ' are those from Lemma 2.8.

Proof. Fix $t \in [0, T]$. Then $\exists j \geq 1$ s.t. $t \in I_j$. Hence (2.44), (2.67), (2.75) and (H2) imply for $p \in (6, \infty)$

$$\begin{aligned} d(\Theta(\cdot, t), \bar{\Theta}(\cdot, t)) &\leq C \left(1 + |F|^p + |Z|^2 + |w|^2 + |V|^2 \right) \\ &\leq C \left(1 + G(\Xi^{j-1}) + G(\Xi^j) + |v^{j-1}|^2 + |v^j|^2 \right) \end{aligned} \quad (2.76)$$

with $C = C(M) > 0$ independent of h, j and t . Hence (2.43) and (2.76) imply

$$\int_{\mathbb{T}^3} d(\Theta(\cdot, t), \bar{\Theta}(\cdot, t)) dx \leq C'(1 + E_0), \quad \forall t \in [0, T] \quad (2.77)$$

for some $C' = C'(M) > 0$. Then (2.70) and (2.77) imply the lemma. \square

Lemma 2.10 (*Q bound*). *There exists $\lambda = \lambda(M) > 0$ such that*

$$|Q(x, t)| \leq \lambda d(\Theta, \bar{\Theta}), \quad (x, t) \in \mathbb{T}^3 \times [0, T] \quad (2.78)$$

where the term Q is defined by (2.52).

Proof. Let $C = C(M) > 0$ be a generic constant. Notice that $\forall F_1, F_2 \in M^{3 \times 3}$

$$|\Psi_{,i\alpha}^A(F_1) - \Psi_{,i\alpha}^A(F_2)| \leq \begin{cases} 0, & A = 1, \dots, 9 \\ |F_1 - F_2|, & A = 10, \dots, 18 \\ 3(|F_1| + |F_2|)|F_1 - F_2|, & A = 19 \end{cases} \quad (2.79)$$

and hence

$$|\Psi_{,i\alpha}^A(F) - \Psi_{,i\alpha}^A(\bar{F})| \leq C(1 + |F|)|F - \bar{F}|, \quad A = 1 \dots 19. \quad (2.80)$$

Then, using (2.75) and (2.80) we estimate the first term of Q :

$$\begin{aligned} & |\partial_\alpha(G_{,A}(\bar{\Xi}))(\Psi_{,i\alpha}^A(F) - \Psi_{,i\alpha}^A(\bar{F}))(V_i - \bar{V}_i)| \\ & \leq C \left((1 + |F|^2)|F - \bar{F}|^2 + |V - \bar{V}|^2 \right). \end{aligned} \quad (2.81)$$

Observe now that (2.72) and (2.79)₁ imply for all $\Xi_1, \Xi_2 \in \mathbb{R}^{22}$, $F_3, F_4 \in \mathbb{R}^9$

$$\begin{aligned} & (G_{,A}(\Xi_1) - G_{,A}(\Xi_2))(\Psi_{,i\alpha}^A(F_3) - \Psi_{,i\alpha}^A(F_4)) \\ & = (R_{,A}(\Xi_1) - R_{,A}(\Xi_2))(\Psi_{,i\alpha}^A(F_3) - \Psi_{,i\alpha}^A(F_4)). \end{aligned} \quad (2.82)$$

Thus, by (H1), (2.80) and (2.82) we obtain the estimate for the second term:

$$\begin{aligned} & |\partial_\alpha \bar{V}_i(G_{,A}(\Xi) - G_{,A}(\bar{\Xi}))(\Psi_{,i\alpha}^A(F) - \Psi_{,i\alpha}^A(\bar{F}))| \\ & \leq C \left(|\Xi - \bar{\Xi}|^2 + (1 + |F|^2)|F - \bar{F}|^2 \right). \end{aligned} \quad (2.83)$$

Finally, we define for each $A = 1, \dots, 19$

$$\begin{aligned} J_A &:= G_{,A}(\Xi) - G_{,A}(\bar{\Xi}) - G_{,AB}(\bar{\Xi})(\Xi - \bar{\Xi})_B \\ &= \int_0^1 \int_0^1 s(\Xi - \bar{\Xi})^T \nabla^2 G_{,A}(\hat{\Xi})(\Xi - \bar{\Xi}) ds d\tau \end{aligned} \quad (2.84)$$

where

$$\hat{\Xi} = (\hat{F}, \hat{Z}, \hat{w}) := \bar{\Xi} + \tau s(\Xi - \bar{\Xi}), \quad \tau, s \in [0, 1].$$

By (2.37) and (H5) we have for each $A = 1, \dots, 19$

$$|(\Xi - \bar{\Xi})^T \nabla^2 G_{,A}(\hat{\Xi})(\Xi - \bar{\Xi})| \leq C \left(|F - \bar{F}|^2 |\hat{F}|^{p-3} + |\Xi - \bar{\Xi}|^2 \right). \quad (2.85)$$

Then by (2.75) and (2.84)-(2.85) we obtain the estimate for the third term:

$$\begin{aligned} & |\partial_\alpha \bar{V}_i \Psi_{,i\alpha}^A(\bar{F}) J_A| \\ & \leq C \left(|\Xi - \bar{\Xi}|^2 + |F - \bar{F}|^2 \int_0^1 \int_0^1 |\bar{F} + \tau s(F - \bar{F})|^{p-3} ds d\tau \right) \\ & \leq C \left(|\Xi - \bar{\Xi}|^2 + |F - \bar{F}|^2 (1 + |F|^{p-3}) \right). \end{aligned} \quad (2.86)$$

Thus by (2.67), (2.81), (2.83) and (2.86) we conclude for $p \in (6, \infty)$

$$|Q(x, t)| \leq C \left(|\Theta - \bar{\Theta}|^2 + (1 + |F|^{p-2}) |F - \bar{F}|^2 \right) \leq C d(\Theta, \bar{\Theta}).$$

□

Next, we set

$$I'_j := I_j \cap [0, T] = [(j-1)h, jh] \cap [0, T], \quad j \geq 1$$

and prove:

Lemma 2.11 (*D^j bound*). *Let D^j be the term defined by (2.53) and $(j-1)h < T$.*

Then

$$D^j \in L^\infty(I'_j; L^1(\mathbb{T}^3)) \quad (2.87)$$

and $\exists C_D > 0$ independent of h, j such that $\forall \tau \in \bar{I}'_j := [(j-1)h, jh] \cap [0, T]$

$$\begin{aligned} & \int_{(j-1)h}^\tau \int_{\mathbb{T}^3} \left(\frac{1}{h} D^j \right) dx dt \\ & \geq a(\tau) C_D \int_{\mathbb{T}^3} |\delta \Theta^j|^2 + (|F^{j-1}|^{p-2} + |F^j|^{p-2}) |\delta F^j|^2 dx \geq 0 \end{aligned} \quad (2.88)$$

with

$$a(\tau) := \frac{\tau - h(j-1)}{h} \in [0, 1], \quad \tau \in \bar{I}'_j. \quad (2.89)$$

Proof. By (H1), (2.29) and the definition of D^j we have for $t \in I'_j$

$$D^j = (v - V) \delta v^j + (\nabla H(f) - \nabla H(F)) \delta F^j + (\nabla R(\xi) - \nabla R(\Xi)) \delta \Xi^j. \quad (2.90)$$

Consider each of the three terms in (2.90). Notice that, by (2.44)-(2.45), we have

$$\begin{aligned} v(\cdot, t) - V(\cdot, t) &= (1 - a(t)) \delta v^j \\ \xi(\cdot, t) - \Xi(\cdot, t) &= (1 - a(t)) \delta \Xi^j. \end{aligned} \quad (2.91)$$

Using (2.91) we compute

$$\begin{aligned} (v - V) \delta v^j &= (1 - a(t)) |\delta v^j|^2 \\ (\nabla R(\xi) - \nabla R(\Xi)) \delta \Xi^j &= (1 - a(t)) \int_0^1 (\delta \Xi^j)^T \nabla^2 R(\hat{\Xi}) (\delta \Xi^j) ds \\ (\nabla H(f) - \nabla H(F)) \delta F^j &= (1 - a(t)) \int_0^1 (\delta F^j)^T \nabla^2 H(\hat{F}) (\delta F^j) ds \end{aligned} \quad (2.92)$$

where

$$\hat{\Xi} = (\hat{F}, \hat{Z}, \hat{w}) := s\xi(\cdot, t) + (1 - s)\Xi(\cdot, t), \quad s \in [0, 1].$$

Then (H1), (2.90) and (2.92) together with the fact that $(1 - a(t)) \in [0, 1]$ imply

$$|D^j(\cdot, t)| \leq \left(|\delta v^j|^2 + \gamma' |\delta \Xi^j|^2 + \kappa' |\delta F^j|^2 \int_0^1 |\hat{F}(s, t)|^{p-2} ds \right). \quad (2.93)$$

Consider now the two latter terms in (2.93). Recalling that $\hat{F} = sf - (1 - s)F$ and using (H2) together with (2.44)-(2.45) we obtain

$$\begin{aligned} &\gamma' |\delta \Xi^j|^2 + \kappa' |\delta F^j|^2 \int_0^1 |\hat{F}(s, t)|^{p-2} ds \\ &\leq C (1 + |F^{j-1}|^p + |F^j|^p + |Z^{j-1}|^2 + |Z^j|^2 + |w^{j-1}|^2 + |w^j|) \end{aligned}$$

for some $C > 0$ independent of h, j and t . Thus, combining the last inequality with (H2), the growth estimate (2.43) and (2.93), we conclude

$$\int_{\mathbb{T}^3} |D^j(x, t)| dx \leq \nu'(1 + E_0), \quad \forall t \in I'_j \quad (2.94)$$

for some $\nu' > 0$ independent of h, j and t . This proves (2.87).

Let us now estimate D^j from below. By (2.90), (2.92) and (H1) we obtain

$$D^j(\cdot, t) \geq \nu(1 - a(t)) \left(|\delta\Theta^j|^2 + |\delta F^j|^2 \int_0^1 |\hat{F}(s, t)|^{p-2} ds \right) \geq 0 \quad (2.95)$$

for $\nu = \min(1, \gamma, \kappa) > 0$. Notice that

$$\hat{F}(s, t) = sf(t) + (1 - s)F(t) = F^j + (1 - s)(1 - a(t))(F^{j-1} - F^j).$$

Then, by making use of Lemma 2.7 we obtain for $\tau \in \bar{I}'_j$

$$\begin{aligned} & \int_{(j-1)h}^{\tau} \left((1 - a(t)) |\delta F^j|^2 \int_0^1 |\hat{F}(s, t)|^{p-2} ds \right) dt \\ &= h |\delta F^j|^2 \int_0^{a(\tau)} \int_0^1 (1 - \beta) |F^j + \alpha(1 - \beta)(F^{j-1} - F^j)|^{p-2} d\alpha d\beta \\ &\geq h a(\tau) c' (|F^{j-1}|^{p-2} + |F^j|^{p-2}) |\delta F^j|^2 \end{aligned}$$

where we used the change of variables $\alpha = 1 - s$ and $\beta = a(t)$. Similarly, we get

$$\int_{(j-1)h}^{\tau} (1 - a(t)) |\delta\Theta^j|^2 dt = h |\delta\Theta^j|^2 \int_0^{a(\tau)} (1 - \beta) d\beta \geq \frac{h a(\tau)}{2} |\delta\Theta^j|^2.$$

Then (2.95) and the last two estimates imply (2.88) for $C_D = \min(\nu c', \frac{\nu}{2}) > 0$. \square

Lemma 2.12 (S bound). *Let S be the term defined by (2.54) and $(j - 1)h < T$.*

Then

$$S \in L^\infty(I'_j; L^1(\mathbb{T}^3)) \quad (2.96)$$

and $\exists C_S > 0$ independent of h, j such that for any $\varepsilon > 0$ and all $\tau \in \bar{I}'_j$

$$\begin{aligned} & \int_{(j-1)h}^{\tau} \int_{\mathbb{T}^3} |S(x, t)| \, dx \, dt \\ & \leq C_S \left[a(\tau)(h + \varepsilon) \int_{\mathbb{T}^3} |\delta \Theta^j|^2 + (|F^{j-1}|^{p-2} + |F^j|^{p-2}) |\delta F^j|^2 \, dx \right. \\ & \quad \left. + \frac{a(\tau)h^2}{\varepsilon} (3 + 2E_0) + \int_{(j-1)h}^{\tau} \int_{\mathbb{T}^3} d(\Theta, \bar{\Theta}) \, dx \, dt \right] \end{aligned} \quad (2.97)$$

with $a(\tau)$ defined by (2.89).

Proof. As before, we let $C = C(M) > 0$ be a generic constant and remind the reader that all estimates are done for $t \in I'_j$.

Observe that (2.44)₂, (2.45)₃ and (2.89) imply

$$F(\cdot, t) - \tilde{f}(\cdot, t) = a(t) \delta F^j.$$

Hence by (2.44)₂, (2.45)₃, (2.79), (2.89) and the identity above we get the estimate

$$\begin{aligned} |\Psi_{,i\alpha}^A(\tilde{f}) - \Psi_{,i\alpha}^A(F)| & \leq C(1 + |\tilde{f}| + |F|) |F - \tilde{f}| \\ & \leq C(1 + |F^{j-1}| + |F^j|) |\delta F^j|. \end{aligned} \quad (2.98)$$

Thus (2.80), (2.89), (2.91)₁, (2.98) and the Young's inequality imply

$$\begin{aligned} & |\Psi_{,i\alpha}^A(\bar{F})(v_i - V_i)| \\ & + |(\Psi_{,i\alpha}^A(F) - \Psi_{,i\alpha}^A(\bar{F}))(v_i - V_i)| \\ & + |(\Psi_{,i\alpha}^A(\tilde{f}) - \Psi_{,i\alpha}^A(F))(v_i - V_i)| \\ & + |(\Psi_{,i\alpha}^A(\tilde{f}) - \Psi_{,i\alpha}^A(F))(V_i - \bar{V}_i)| \\ & \leq C \left(|\delta v^j| + (1 + |F|^2) |F - \bar{F}|^2 + |\delta v^j|^2 \right. \\ & \quad \left. + (1 + |F^{j-1}|^2 + |F^j|^2) |\delta F^j|^2 + |V - \bar{V}|^2 \right). \end{aligned} \quad (2.99)$$

We also notice that for all $F_1, F_2 \in M^{3 \times 3}$

$$H_{,i\alpha}(F_1) - H_{,i\alpha}(F_2) = \int_0^1 \frac{\partial^2 H}{\partial F_{i\alpha} \partial F_{lm}} (sF_1 + (1-s)F_2) (F_1 - F_2)_{lm} ds.$$

Hence (H1), (H5), (2.89), (2.91)₂ and the identity above imply

$$\begin{aligned} & |\Psi_{,i\alpha}^A(\bar{F}) (G_{,A}(\xi) - G_{,A}(\Xi))| \\ & \leq C(|\nabla H(f) - \nabla H(F)| + |\nabla R(\xi) - \nabla R(\Xi)|) \\ & \leq C\left(|f - F| \int_0^1 |sf + (1-s)F|^{p-2} ds + |\xi - \Xi|\right) \\ & \leq C\left((|F^{j-1}|^{p-2} + |F^j|^{p-2})|\delta F^j| + |\delta \Xi^j|\right). \end{aligned} \tag{2.100}$$

Next, by (H1), (2.80), (2.82), (2.89), (2.91)₂ and (2.98) we obtain

$$\begin{aligned} & |(G_{,A}(\xi) - G_{,A}(\Xi))(\Psi_{,i\alpha}^A(\tilde{f}) - \Psi_{,i\alpha}^A(F))| \\ & + |(G_{,A}(\xi) - G_{,A}(\Xi))(\Psi_{,i\alpha}^A(F) - \Psi_{,i\alpha}^A(\bar{F}))| \\ & + |(G_{,A}(\Xi) - G_{,A}(\bar{\Xi}))(\Psi_{,i\alpha}^A(\tilde{f}) - \Psi_{,i\alpha}^A(F))| \\ & \leq C\left(|\delta \Xi^j|^2 + (1 + |F^{j-1}|^2 + |F^j|^2)|\delta F^j|^2 \right. \\ & \quad \left. + (1 + |F|^2)|F - \bar{F}|^2 + |\Xi - \bar{\Xi}|^2\right). \end{aligned} \tag{2.101}$$

Finally, (2.54), (2.75), and the estimates (2.99)-(2.101) imply for $p \in (6, \infty)$

$$\begin{aligned} |S(\cdot, t)| & \leq C_S \left[(|F^{j-1}|^{p-2} + |F^j|^{p-2})|\delta F^j|^2 + |\delta \Theta^j|^2 \right. \\ & \quad \left. + (|F^{j-1}|^{p-2} + |F^j|^{p-2})|\delta F^j| + |\delta \Theta^j| + d(\Theta, \bar{\Theta}) \right] \end{aligned} \tag{2.102}$$

for some $C_S > 0$ independent of h, j and t . Then, by (2.43) and (2.76) we conclude that the right-hand side of (2.102) is in $L^\infty(I'_j; L^1(\mathbb{T}^3))$ which proves (2.96).

We now pick any $\varepsilon > 0$. Then, employing the Young's inequality, we obtain

$$\begin{aligned} (|F^{j-1}|^{p-2} + |F^j|^{p-2})|\delta F^j| & \leq \frac{h}{\varepsilon} (|F^{j-1}|^{p-2} + |F^j|^{p-2}) \\ & \quad + \frac{\varepsilon}{h} (|F^{j-1}|^{p-2} + |F^j|^{p-2}) |\delta F^j|^2 \end{aligned}$$

and, similarly, $|\delta\Theta^j| \leq \frac{h}{\varepsilon} + \frac{\varepsilon}{h}|\delta\Theta^j|^2$. Thus (2.102) and the last two estimates imply

$$|S(\cdot, t)| \leq C_S \left[\left(1 + \frac{\varepsilon}{h}\right) (|\delta\Theta^j|^2 + (|F^{j-1}|^{p-2} + |F^j|^{p-2})|\delta F^j|^2) + \frac{h}{\varepsilon} (1 + |F^{j-1}|^{p-2} + |F^j|^{p-2}) + d(\Theta, \bar{\Theta}) \right]. \quad (2.103)$$

To this end, we integrate (2.102) and use (H2) along with (2.43) to get (2.96). \square

Gronwall's inequality. We now estimate the left hand side of the relative entropy identity (2.51):

Lemma 2.13 (LHS estimate). *Let η^r , q^r be the relative entropy and relative entropy flux, respectively, defined by (2.46) and (2.47). Then*

$$(\partial_t \eta^r - \operatorname{div} q^r) \in L^\infty([0, T], L^1(\mathbb{T}^3)) \quad (2.104)$$

and $\exists \bar{\varepsilon} > 0$ such that for all $h \in (0, \bar{\varepsilon})$ and $\tau \in [0, T]$

$$\int_0^\tau \int_{\mathbb{T}^3} (\partial_t \eta^r - \operatorname{div} q^r) dx dt \leq C_I \left(\tau h + \int_0^\tau \int_{\mathbb{T}^3} d(\Theta, \bar{\Theta}) dx dt \right). \quad (2.105)$$

for some constant $C_I = C_I(M, E_0, \bar{\varepsilon}) > 0$.

Proof. Lemma 2.8, (2.78), (2.87), and (2.96) imply that the right-hand side of the relative entropy identity (2.51) is in $L^\infty([0, T]; L^1(\mathbb{T}^3))$. This proves (2.104).

Notice that the constants C_D and C_S (that appear in Lemmas 2.11 and 2.12, respectively) are independent of h, j . Then set $\bar{\varepsilon} := C_D/(2C_S)$. Take now $h \in (0, \bar{\varepsilon})$ and $\tau \in [0, T]$. Using Lemmas 2.10, 2.11 and 2.12 (with $\varepsilon = \bar{\varepsilon}$) along with the fact that $-C_D + C_S(h + \bar{\varepsilon}) \leq 0$ we get

$$\int_0^\tau \int_{\mathbb{T}^3} \left(-\frac{1}{h} \sum_{j=1}^\infty \mathcal{X}^j(t) D^j + |S| + |Q| \right) dx dt \leq C_I \left(\tau h + \int_0^\tau \int_{\mathbb{T}^3} d(\Theta, \bar{\Theta}) dx dt \right)$$

with $C_I := 3 \max(C_S(1 + E_0)/\bar{\varepsilon}, C_S + \lambda) > 0$. Hence by (2.51) and the estimate above we obtain (2.105). \square

Observe that (P4)-(P5), (2.47), (2.55), (2.59)-(2.60) and (2.62) imply

$$\operatorname{div} q^r \in L^\infty([0, T]; L^1(\mathbb{T}^3)) \quad (2.106)$$

and hence by (2.104)

$$\partial_t \eta^r \in L^\infty([0, T]; L^1(\mathbb{T}^3)). \quad (2.107)$$

Take now arbitrary $h \in (0, \bar{\varepsilon})$ and $\tau \in [0, T]$. Due to periodic boundary conditions (by the density argument) we have $\int_{\mathbb{T}^3} (\operatorname{div} q^r(x, s)) dx = 0$ for a.e. $s \in [0, T]$ and hence

$$\int_0^\tau \int_{\mathbb{T}^3} \operatorname{div} q^r dx dt = 0.$$

Finally, by construction for each fixed $\bar{x} \in \mathbb{T}^3$ the function $\eta^r(\bar{x}, t) : [0, T] \rightarrow \mathbb{R}$ is absolutely continuous with the weak derivative $\partial_t \eta^r(\bar{x}, t)$. Then, by (2.107) and the Fubini's theorem we have

$$\int_0^\tau \int_{\mathbb{T}^3} \partial_t \eta^r dx dt = \int_{\mathbb{T}^3} \left[\int_0^\tau \partial_t \eta^r(x, t) d\tau \right] dx = \int_{\mathbb{T}^3} (\eta^r(x, \tau) - \eta^r(x, 0)) dx.$$

Thus by Lemma 2.9, (2.104)-(2.107) and the two identities above we obtain

$$\mathcal{E}(\tau) \leq \bar{C} \left(\mathcal{E}(0) + \int_0^\tau \mathcal{E}(t) dt + h \right) \quad (2.108)$$

with $\bar{C} := \frac{T}{\mu} \max(C_I, \mu')$ independent of τ, h . Since $\tau \in [0, T]$ is arbitrary, by (2.108) and the Gronwall's inequality we conclude

$$\mathcal{E}(\tau) \leq \bar{C} (\mathcal{E}(0) + h) e^{\bar{C}\tau}, \quad \forall \tau \in [0, T].$$

In this case, if $\mathcal{E}^{(h)}(0) \rightarrow 0$ as $h \downarrow 0$, then $\sup_{\tau \in [0, T]} (\mathcal{E}^{(h)}(\tau)) \rightarrow 0$, as $h \downarrow 0$.

Chapter 3

A variational approximation scheme for radial elasticity that preserves the positivity of Jacobians

The purpose of this chapter is to present a variational approximation scheme for the radial elasticity which preserves the positivity of the quantity $\det \nabla y$, necessary to interpret y as a physically realizable motion.

The major parts of the chapter (besides few elementary lemmas and theorems) were first published in "*A Variational approximation scheme for radial polyconvex elasticity that preserves the positivity of Jacobians*" in Volume 10, Issue 1 (2012), published by International Press[©].

3.1 Background Information

3.1.1 Radial Isotropic elasticity

We consider the equations of nonlinear elasticity

$$y_{tt} = \operatorname{div} S(\nabla y) \quad \text{in } \mathcal{B} \times [0, \infty) \quad (3.1)$$

on the unit ball $\mathcal{B} = \{x \in \mathbb{R}^n : |x| < 1\}$, subject to uniform stretching at the boundary

$$y(x, t) = \lambda x, \quad (x, t) \in \partial\mathcal{B} \times [0, \infty) \quad (3.2)$$

and initial conditions

$$y(x, 0) = w_0(R) \frac{x}{R}, \quad y_t(x, 0) = v_0(R) \frac{x}{R}, \quad x \in \mathcal{B} \setminus \{0\} \quad (3.3)$$

where $R := |x|$ and $w_0 : [0, 1) \rightarrow \mathbb{R}$ is nonnegative.

We employ the constitutive theory of *hyperelasticity* which postulates that the Piola-Kirchhoff stress tensor S is expressed as the gradient,

$$S(F) = \frac{\partial W}{\partial F}(F), \quad (3.4)$$

of the stored-energy function $W : M_+^{3 \times 3} \rightarrow \mathbb{R}$ of the elastic body. In addition, we assume that the elastic material is *isotropic*, i.e. for all proper rotations $Q \in SO(3)$

$$W(FQ) = W(F), \quad \forall F \in M_+^{3 \times 3}. \quad (3.5)$$

We seek for those solutions of (3.1) which correspond to physically realizable motions. Therefore we assume that the stored energy is *frame-indifferent*, i.e. for all proper rotations $Q \in SO(3)$

$$W(QF) = W(F), \quad \forall F \in M_+^{3 \times 3} \quad (3.6)$$

and to exclude interpenetration of matter we require

$$\det \nabla y > 0 \quad \text{a.e. } x \in \mathcal{B}. \quad (3.7)$$

Definition. Let $F \in M_+^{3 \times 3}$. The eigenvalues v_1, v_2, v_3 of the matrix $(F^T F)^{1/2}$ are called the singular values or principal stretches of F .

Remark 3.1. Notice that $F \in M_+^{3 \times 3}$ implies that the matrix $F^T F$ is symmetric positive definite. Thus $(F^T F)^{1/2}$ in the definition above stands for the principal

square root of $F^T F$ and is (by itself) symmetric positive definite. Thus, the principal stretches v_1, v_2, v_3 for the matrix $F \in M_+^{3 \times 3}$ are always positive.

We next show that the stored energy W , for isotropic hyperelastic materials, can be expressed as a *symmetric* function of the principal stretches (see the representation theorems for isotropic functions in [2, p.472] and [31, p.317]). That is

$$W(F) = \Phi(v_1, v_2, v_3), \quad \forall F \in M_+^{n \times n} \quad (3.8)$$

where v_1, \dots, v_n are the singular values of F and

$$\Phi(v_1, v_2, v_3) : \mathbb{R}_+^3 = \{v \in \mathbb{R}^3 : v_i > 0 \ i = 1, 2, 3\} \rightarrow \mathbb{R}$$

is a symmetric function. Indeed, by the polar decomposition theorem any matrix $F \in M_+^{3 \times 3}$ is expressed in the form $F = RU$ with $R \in SO(3)$ and symmetric positive definite $U = (F^T F)^{1/2}$. By definition, v_1, v_2, v_3 are eigenvalues of U . Thus

$$U = Q \operatorname{diag}(v_1, v_2, v_3) Q^T$$

where Q is the orthogonal matrix of eigenvectors. Thus, employing properties of isotropy (3.5) and frame-indifference (3.6), we have

$$W(QFQ^T) = W(F), \quad \forall F \in M_+^{3 \times 3}, Q \in SO(3). \quad (3.9)$$

and hence

$$W(F) = W(RU) = W(\operatorname{diag}(v_1, v_2, v_3)) =: \Phi(v_1, v_2, v_3). \quad (3.10)$$

Furthermore, for each permutation $\pi : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ we clearly have

$$U = Q_\pi \operatorname{diag}(v_{\pi(1)}, v_{\pi(2)}, v_{\pi(3)}) Q_\pi^T$$

where the orthogonal matrix Q_π is obtained via the (corresponding to π) permutation of the columns of the matrix Q . Hence, using (3.9), we conclude that

$$\Phi(v_1, v_2, v_3) = \Phi(v_{\pi(1)}, v_{\pi(2)}, v_{\pi(3)})$$

and this proves that Φ is symmetric.

Observe also that due to the symmetry of Φ we have

$$\frac{\partial \Phi}{\partial v_i}(v_1, v_2, v_3) = \frac{\partial \Phi}{\partial v_{\pi(i)}}(v_{\pi(1)}, v_{\pi(2)}, v_{\pi(3)}), \quad i = 1, 2, 3. \quad (3.11)$$

Definition. A function $f : \mathcal{B} \setminus \{0\} \rightarrow \mathbb{R}^n$ is called *radial* if

$$f(x) = w(R) \frac{x}{R}, \quad w : [0, \infty) \rightarrow [0, \infty), \quad R := |x|.$$

Our next goal is to recast the problem (3.1) for radial solutions. Before we proceed we will prove several lemmas used in our further calculations.

Lemma 3.1. *Let $A \in M^{n \times n}$ be defined by*

$$A = \alpha I + \beta \frac{z \otimes z}{|z|^2} \quad (3.12)$$

where $z \neq 0 \in \mathbb{R}^n$ and α and β are real numbers. Let $Q \in SO(n)$ be a proper rotation satisfying $Qe_1 = \frac{z}{|z|}$. Let q_j denote the j -th column of Q for $j = 1, \dots, n$.

Then vectors $\{q_1, \dots, q_n\}$ are eigenvectors of A with the corresponding eigenvalues $\lambda_1 = \alpha + \beta$ and $\lambda_2 = \dots = \lambda_n = \alpha$ and hence

$$A = QDQ^T \quad \text{with} \quad D := \text{diag}(\alpha + \beta, \alpha, \alpha, \dots, \alpha).$$

Proof. By assumption $Qe_1 = z/|z|$. Hence $q_1 = z/|z|$ and

$$Aq_1 = \alpha q_1 + \beta \frac{\langle z, q_1 \rangle z}{|z|^2} = \alpha q_1 + \beta \frac{z}{|z|} = (\alpha + \beta)q_1.$$

As $Q \in SO(n)$, we have $Q^T Q = I$ and hence $\langle q_1, q_j \rangle = 0$ for $j = 2, \dots, n$. Thus

$$(z \otimes z)q_j = \langle z, q_j \rangle z = \langle q_1, q_j \rangle |z| = 0, \quad j = 2, \dots, n$$

and we conclude that

$$Aq_j = \alpha q_j \quad \text{for } j = 2, \dots, n.$$

Finally, $Q \in SO(n)$ implies $\det Q \neq 0$ and $Q^{-1} = Q^T$. Hence

$$A = QDQ^T \quad \text{with } D = \text{diag}(\alpha + \beta, \alpha, \alpha, \dots, \alpha).$$

□

Lemma 3.2. *Let $z \neq 0 \in \mathbb{R}^n$ be given. Let γ and δ be some real numbers, $Q \in SO(n)$ be a proper rotation satisfying*

$$Qe_1 = \frac{z}{|z|}$$

and $D := \text{diag}(\gamma, \delta, \delta, \dots, \delta)$. Then

$$QDQ^T = \delta I + (\gamma - \delta) \frac{z \otimes z}{|z|^2}.$$

Proof. Notice that

$$D = \gamma [e_1 \otimes e_1] + \delta [I - e_1 \otimes e_1] = \delta I + (\gamma - \delta) e_1 \otimes e_1.$$

Hence

$$\begin{aligned} QDQ^T &= Q [\delta I + (\gamma - \delta) e_1 \otimes e_1] Q^T = \\ &\delta I + (\gamma - \delta) [(Qe_1) \otimes (Qe_1)] = \delta I + (\gamma - \delta) \frac{x \otimes x}{|x|^2} \end{aligned}$$

□

Lemma 3.3. *Let $\tilde{W} : M_+^{n \times n} \rightarrow \mathbb{R}^n$ be smooth, isotropic, and frame-indifferent and $\tilde{\Phi} : \mathbb{R}_{++}^n \rightarrow \mathbb{R}$ be a symmetric function such that*

$$\tilde{W}(F) = \tilde{\Phi}(v_1, \dots, v_n), \quad \forall F \in M_+^{n \times n}$$

where v_1, \dots, v_n are eigenvalues of $(F^T F)^{\frac{1}{2}}$. Assume that

$$\bar{F} = \text{diag}(v_1, \dots, v_n) \quad \text{with} \quad v_i > 0, \quad i = 1, \dots, n.$$

Then

$$\frac{\partial \tilde{W}}{\partial F}(\bar{F}) = \text{diag} \left(\tilde{\Phi}_{,1}(V), \dots, \tilde{\Phi}_{,n}(V) \right)$$

where

$$V := (v_1, \dots, v_n) \quad \text{and} \quad \tilde{\Phi}_{,i}(V) := \frac{\partial \tilde{\Phi}}{\partial v_i}(v_1, \dots, v_n).$$

Proof. By assumption $\bar{F} \in M_+^{n \times n}$. Then it is easy to see that for all $h \in \mathbb{R}$ satisfying

$$0 < |h| < \min_{i=1, \dots, n} (v_i)$$

the matrix $[\bar{F} + h e_i \otimes e_j] \in M_+^{n \times n}$. In this case, for each $i = 1, \dots, n$ we have

$$\tilde{W}(\bar{F} + h e_i \otimes e_i) = \tilde{\Phi}(v_1, \dots, v_i + h, \dots, v_n)$$

and hence

$$\begin{aligned} \frac{\partial \tilde{W}}{\partial F_{ii}}(\bar{F}) &= \lim_{h \rightarrow 0} \frac{\tilde{W}(\bar{F} + h e_i \otimes e_i) - \tilde{W}(\bar{F})}{h} \\ &= \lim_{h \rightarrow 0} \frac{\tilde{\Phi}(v_1, \dots, v_i + h, \dots, v_n) - \tilde{\Phi}(v_1, \dots, v_n)}{h} = \tilde{\Phi}_{,i}(V). \end{aligned}$$

Next, observe that for each $i, j = 1, \dots, n$ with $i \neq j$

$$\tilde{W}(\bar{F} + h e_i \otimes e_j) = \tilde{\Phi}(\tilde{v}_1(h), \dots, \tilde{v}_n(h))$$

where

$$\begin{aligned}\tilde{v}_i(h) &= \left(\frac{1}{2} \left[v_i^2 + v_j^2 + h^2 + \sqrt{(v_i^2 + v_j^2 + h^2)^2 - 4v_i^2 v_j^2} \right] \right)^{1/2}, \\ \tilde{v}_j(h) &= \left(\frac{1}{2} \left[v_i^2 + v_j^2 + h^2 - \sqrt{(v_i^2 + v_j^2 + h^2)^2 - 4v_i^2 v_j^2} \right] \right)^{1/2} \\ \tilde{v}_k(h) &= v_k, \quad k \notin \{i, j\}.\end{aligned}$$

Hence, by direct calculation, we obtain

$$\frac{\partial \tilde{W}}{\partial F_{ij}}(\bar{F}) = 0, \quad i \neq j$$

and this finishes the proof. □

Lemma 3.4. *Let $P : \mathbb{R}^n \setminus \{0\} \rightarrow M^{n \times n}$ be defined by*

$$P(z) = K(|z|)I + B(|z|)\frac{z \otimes z}{|z|^2}, \quad z \in \mathbb{R}^n$$

where K and B are smooth scalar functions. Then for all $z \neq 0$

$$\operatorname{div} P(z) = \left(K'(|z|) + B'(|z|) + (n-1)\frac{B(|z|)}{|z|} \right) \frac{z}{|z|}.$$

Proof. For each $i \in \{1, \dots, n\}$ we have

$$\begin{aligned}(\operatorname{div} P(z))_i &= \sum_{j=1}^n \partial_{z_j} \left(K \delta_{ij} + B \frac{z_i z_j}{|z|^2} \right) \\ &= \partial_{z_i} K + \sum_{\substack{j=1 \\ j \neq i}}^n \left(\partial_{z_j} B \frac{z_i z_j}{|z|^2} + B z_i \left(\frac{1}{|z|^2} - \frac{2z_j^2}{|z|^4} \right) \right) \\ &\quad + \left(\partial_{z_i} B \frac{z_i z_i}{|z|^2} + B z_i \left[\frac{2}{|z|^2} - \frac{2z_i^2}{|z|^4} \right] \right) \\ &= \partial_{z_i} K + \sum_{j=1}^n \left(\partial_{z_j} B \frac{z_i z_j}{|z|^2} \right) + (n-1) \frac{B z_i}{|z|^2} \\ &= \left(K' + B' + (n-1) \frac{B(|z|)}{|z|} \right) \frac{z_i}{|z|}.\end{aligned}$$

□

At this point we are ready to transform equations (3.1) into the equations that monitor the evolution of the magnitude of the radial solution to (3.1). To avoid inessential technicalities we will perform a series of calculations for the case when the solution y is smooth.

Theorem 3.1. *Let y be a smooth solution to (3.1) for $t \in [0, T]$. Assume that y satisfies (3.2), (3.3) and the constraint (3.7) and the constitutive hypotheses (3.4)-(3.6) hold. Finally, assume that y is radial and has the form*

$$y(x, t) = w(R, t) \frac{x}{R} \quad \text{for } x \neq 0 \quad (3.13)$$

with $w : [0, 1) \times [0, T] \rightarrow \mathbb{R}$ satisfying $w(R, t) \geq 0$. Then $\forall x \in \mathcal{B} \setminus \{0\}$ we have:

(R1) *The deformation gradient of y is expressed by*

$$\nabla y = \frac{w}{R} I + \left(w_R(R, t) - \frac{w}{R} \right) \frac{x \otimes x}{R^2}. \quad (3.14)$$

(R2) *The condition (3.7) expressing the requirement that matter cannot penetrate itself transforms into*

$$\det \nabla y = w_R \left(\frac{w}{R} \right)^2 > 0, \quad R \in (0, 1) \quad (3.15)$$

and dictates $w_R, \frac{w}{R} > 0$.

(R3) *The principal stretches v_1, v_2, v_3 of the deformation gradient ∇y satisfy*

$$v_1 = w_R, \quad v_2 = v_3 = \frac{w}{R}$$

in which case the stored energy of the deformation is given by

$$W(\nabla y) = \Phi(w_R, \frac{w}{R}, \frac{w}{R})$$

and the first Piola-Kirchhoff stress by

$$S(\nabla y) = \frac{\partial \Phi}{\partial v_2} \left(w_R, \frac{w}{R}, \frac{w}{R} \right) I + \left[\frac{\partial \Phi}{\partial v_1} \left(w_R, \frac{w}{R}, \frac{w}{R} \right) - \frac{\partial \Phi}{\partial v_2} \left(w_R, \frac{w}{R}, \frac{w}{R} \right) \right] \frac{x \otimes x}{R^2}.$$

(R4) The amplitude $w(R, t) = |y(x, t)|$ satisfies the second order equation

$$R^2 \partial_{tt} w = \frac{\partial}{\partial R} \left(R^2 \frac{\partial \Phi}{\partial v_1} \left(w_R, \frac{w}{R}, \frac{w}{R} \right) \right) - R \sum_{i=2}^3 \frac{\partial \Phi}{\partial v_i} \left(w_R, \frac{w}{R}, \frac{w}{R} \right) \quad (3.16)$$

for $R \in (0, 1)$. In addition, boundary and initial conditions (3.2), (3.3) imply

$$w(1, t) = \lambda \quad \text{and} \quad w(R, 0) = w_0(R), \quad w_t(R, 0) = v_0(R), \quad R \in (0, 1), t \in [0, T]$$

Proof. The property (R1) follows from the direct computations of the gradient for $y = w(R, t) \frac{x}{R}$. Now, observe that by (R1) the gradient ∇y is of the same form as the matrix A in Lemma 3.1. Then, applying the lemma to the matrix ∇y , we conclude that for each $x \in \mathcal{B} \setminus \{0\}$

$$\nabla y(x, t) = \bar{Q} \operatorname{diag} \left(w_R, \frac{w}{R}, \frac{w}{R} \right) \bar{Q}^T \quad (3.17)$$

where $\bar{Q} = \bar{Q}(x) \in SO(3)$ is a proper rotation satisfying $\bar{Q}(x)e_1 = \frac{x}{R}$. Hence by (3.7) and (3.17) we must have

$$\det \nabla y = w_R \left(\frac{w}{R} \right)^2 > 0, \quad R \in (0, 1)$$

Then $w_R > 0$ for $R \in (0, 1)$ and therefore for each $t \in [0, T]$ the function $w(R, t)$ is a strictly increasing function of R . By assumption $w(0, t) \geq 0$ and hence $w(R, t) > 0$ for all $R \in (0, 1)$. This proves (R2).

Next, notice that by (R1), (R2) the matrix $\nabla y \in M_+^{3 \times 3}$ is symmetric positive definite and hence we clearly have

$$\nabla y = (\nabla y^T \nabla y)^{1/2}$$

which implies that the singular values and eigenvalues for ∇y coincide. Hence by (3.17) we conclude that singular values of ∇y are

$$v_1 = w_R, \quad v_2 = v_3 = \frac{w}{R}.$$

Hence by (3.8) the stored energy of the deformation is given by

$$W(\nabla y) = \Phi(w_R, \frac{w}{R}, \frac{w}{R}).$$

Next, we differentiate the equality (3.9) with respect to F and obtain

$$S(QFQ^T) = QS(F)Q^T, \quad \forall F \in M_+^{3 \times 3}, Q \in SO(3). \quad (3.18)$$

Since $\nabla y \in M_+^{3 \times 3}$, relations (3.17) and (3.18) imply

$$S(\nabla y(x, t)) = \bar{Q} S(\text{diag}(w_R, \frac{w}{R}, \frac{w}{R})) \bar{Q}^T. \quad (3.19)$$

Furthermore, using Lemma 3.3 and the fact that $w_R, \frac{w}{R} > 0$, we obtain

$$S(\text{diag}(w_R, \frac{w}{R}, \frac{w}{R})) = \text{diag}(\Phi_{,1}(V), \Phi_{,2}(V), \Phi_{,3}(V)) \quad (3.20)$$

where

$$V := (w_R, \frac{w}{R}, \frac{w}{R}) \quad \text{and} \quad \Phi_{,i}(V) := \frac{\partial \Phi}{\partial v_i}(w_R, \frac{w}{R}, \frac{w}{R}). \quad (3.21)$$

Notice also that the property (3.11) implies

$$\frac{\partial \Phi}{\partial v_2}(a, b, b) = \frac{\partial \Phi}{\partial v_3}(a, b, b), \quad \forall a, b \in \mathbb{R}_+. \quad (3.22)$$

Hence (3.19)-(3.22) and Lemma 3.2 imply

$$S(\nabla y) = \Phi_{,2}(V)I + [\Phi_{,1}(V) - \Phi_{,2}(V)] \frac{x \otimes x}{R^2} \quad (3.23)$$

which finishes the proof of (R3).

Next, by (3.21), (3.23) and Lemma 3.4 we have

$$\operatorname{div} S(\nabla y) = \left(\frac{\partial}{\partial R} (\Phi_{,1}(v)) + 2 \frac{\Phi_{,1}(v) - \Phi_{,2}(v)}{R} \right) \frac{x}{R}. \quad (3.24)$$

Thus by (3.1), (3.22), (3.24) and the fact that $y_{tt} = w_{tt}(R, t) \frac{x}{R}$ we get

$$R^2 \partial_{tt} w = \frac{\partial}{\partial R} \left(R^2 \frac{\partial \Phi}{\partial v_1} \left(w_R, \frac{w}{R}, \frac{w}{R} \right) \right) - R \sum_{i=2}^3 \frac{\partial \Phi}{\partial v_i} \left(w_R, \frac{w}{R}, \frac{w}{R} \right).$$

Finally, notice that $|x| = 1$ whenever $x \in \partial \mathcal{B}$. Hence using (3.2) we get

$$w(1, t) = \lambda, \quad t \in [0, T] \quad (3.25)$$

Also, by (3.3) we have that $w(R, 0) \frac{x}{R} = w_0(R) \frac{x}{R}$ and $w_t(R, 0) \frac{x}{R} = v_0(R) \frac{x}{R}$ for each $x \in \mathcal{B} \setminus \{0\}$. Hence

$$w(R, 0) = w_0(R), \quad w_t(R, 0) = v_0(R), \quad R \in (0, 1), t \in [0, T]$$

and this finishes the proof. □

3.1.2 Polyconvexity in the Radial Case

One possible way to accommodate the condition (3.7) is to let the stored energy W increase without bound as $\det F \rightarrow 0^+$ so that compression of a finite volume down to a point would cost infinite energy. Such behavior would be inconsistent with simultaneously requiring convexity and invariance of the stored energy under rotations. Thus, convexity of W is not a natural assumption. As an alternative, we assume that the stored energy W is *polyconvex*, that is

$$W(F) = \bar{\bar{G}}(F, \operatorname{cof} F, \det F) \quad (3.26)$$

for some *convex* function $\bar{\bar{G}} : M_+^{3 \times 3} \times M_+^{3 \times 3} \times \mathbb{R}_+ \rightarrow \mathbb{R}$.

By assumption, the stored energy W satisfies (3.5), (3.6), and (3.26). Thus there exists convex function $\bar{G} : \mathbb{R}^7 \rightarrow \mathbb{R}$ such that

$$W(F) = \bar{G}(v_1, v_2, v_3, v_2 v_3, v_1 v_3, v_1 v_2, v_1 v_2 v_3), \quad F \in M_+^{3 \times 3} \quad (3.27)$$

where v_1, v_2, v_3 are the singular values of F . Indeed, using the polar decomposition theorem we write $F \in M_+^{3 \times 3}$ in the form $F = RU$ with $R \in SO(3)$ and symmetric positive definite $U = (F^T F)^{1/2}$. By definition, v_1, v_2, v_3 are eigenvalues of U and so

$$U = Q \operatorname{diag}(v_1, v_2, v_3) Q^T$$

where Q is the orthogonal matrix of eigenvectors. Then by (3.9) we get

$$\begin{aligned} W(F) &= W(RU) = W(\operatorname{diag}(v_1, v_2, v_3)) \\ &= \bar{\bar{G}}(\operatorname{diag}(v_1, v_2, v_3), \operatorname{diag}(v_2 v_3, v_1 v_3, v_1 v_2), v_1 v_2 v_3) \\ &=: \bar{G}(v_1, v_2, v_3, v_2 v_3, v_1 v_3, v_1 v_2, v_1 v_2 v_3). \end{aligned}$$

Since $\bar{\bar{G}}$ is convex, we conclude that $\bar{G} : \mathbb{R}^7 \rightarrow \mathbb{R}$ is convex as well.

For now, to avoid technicalities, we assume that

$$y(x, t) = w(R, t) \frac{x}{R}, \quad R \neq 0,$$

with $w(R, t) \geq 0$, is a smooth radial solution to the system (3.1) that satisfies the constraint (3.7). Then, by Theorem 3.1, the singular values are $v_1 = w_R$, $v_2 = v_3 = \frac{w}{R}$ and, for reasons related to the null-Lagrangian structure of an associated variational problem (outlined in the following section), the stored energy could be

expressed in the form

$$\begin{aligned}
W(\nabla y) &= \Phi \left(w_R, \frac{w}{R}, \frac{w}{R} \right) \\
&= \bar{G} \left(w_R, \frac{w}{R}, \frac{w}{R}, \left(\frac{w}{R} \right)^2, w_R \left(\frac{w}{R} \right), w_R \left(\frac{w}{R} \right), w_R \left(\frac{w}{R} \right)^2 \right) \\
&= G \left(\Omega \left(\left(w_R, \frac{w}{R}, \frac{w}{R} \right); R \right); R \right)
\end{aligned} \tag{3.28}$$

where Ω and G are inhomogeneous functions defined by

$$\Omega(V; R) := \left(v_1, v_2, v_3, v_2 v_3 R, v_1 v_3 R, v_1 v_2 R, v_1 v_2 v_3 R^2 \right), \tag{3.29}$$

$$G(\Xi; R) := \bar{G} \left(\xi_1, \xi_2, \xi_3, \xi_4/R, \xi_5/R, \xi_6/R, \xi_7/R^2 \right), \tag{3.30}$$

$V = (v_i)_{i=1\dots 3} \in \mathbb{R}^3$ and $\Xi = (\xi)_{i=1\dots 7} \in \mathbb{R}^7$. The convexity hypothesis on \bar{G} implies that $G(\Xi; R)$ is convex as a function of $\Xi \in \mathbb{R}^7$. In summary,

$$W(\nabla y) = \Phi \left(w_R, \frac{w}{R}, \frac{w}{R} \right) = G(\Omega(\Gamma; R); R), \tag{3.31}$$

$$\text{where } \Gamma := \left(w_R, \frac{w}{R}, \frac{w}{R} \right). \tag{3.32}$$

For simplicity of notation, we henceforth suppress the dependence on R and write $\Omega(V) = \Omega(V; R)$ and $G(\Xi) = G(\Xi; R)$.

By Theorem 3.1, the magnitude $w = |y|$ satisfies the equation (3.16). Thus, taking into account polyconvexity of W , the equation (3.16) can be expressed in the form of the second-order system

$$\begin{aligned}
R^2 \partial_t v &= \frac{\partial}{\partial R} \left(R^2 \Phi_{,1} \left(w_R, \frac{w}{R}, \frac{w}{R} \right) \right) - R \left(\Phi_{,2} + \Phi_{,3} \right) \left(w_R, \frac{w}{R}, \frac{w}{R} \right) \\
\partial_t w &= v
\end{aligned} \tag{3.33}$$

where we use the notation

$$\Phi_{,i}(v_1, v_2, v_3) := \frac{\partial \Phi}{\partial v_i}(v_1, v_2, v_3). \tag{3.34}$$

The equation (3.33) formally satisfies the conservation of mechanical energy identity

$$\partial_t \left(R^2 \left(\frac{v^2}{2} + \Phi \left(w_R, \frac{w}{R}, \frac{w}{R} \right) \right) \right) = \partial_R \left(R^2 v \Phi_{,1} \left(w_R, \frac{w}{R}, \frac{w}{R} \right) \right). \quad (3.35)$$

The mechanical energy and the associated energy flux provide an entropy-entropy flux pair for (3.33) but the entropy is not in general convex. In view of (3.31)–(3.32), $\Phi_{,j}$, $j = 1, \dots, 3$ are expressed as

$$\Phi_{,j}(v_1, v_2, v_3) = \frac{\partial G}{\partial v_j}(\Omega(V)) = \frac{\partial G}{\partial \xi_i}(\Omega(V)) \frac{\partial \Omega^i}{\partial v_j}(V),$$

and (3.33) is written as

$$\begin{aligned} R^2 \partial_t v &= \partial_R \left(R^2 \frac{\partial G}{\partial \xi_i}(\Omega(\Gamma)) \frac{\partial \Omega^i}{\partial v_1}(\Gamma) \right) \\ &\quad - R \frac{\partial G}{\partial \xi_i}(\Omega(\Gamma)) \left(\frac{\partial \Omega^i}{\partial v_2}(\Gamma) + \frac{\partial \Omega^i}{\partial v_3}(\Gamma) \right) \end{aligned} \quad (3.36)$$

$$\partial_t w = v.$$

3.2 Null-Lagrangians and Extensions of Polyconvex Radial Elasticity

3.2.1 Null-Lagrangians in the Radial Case

An alternative approach to derive (3.35) proceeds by considering the extrema of the action functional

$$J[y] = \int_0^T \int_0^1 \left(\frac{1}{2} w_t^2 - \Phi \left(w_R, \frac{w}{R}, \frac{w}{R} \right) \right) R^2 dR dt$$

and deriving (3.16) as the associated Euler-Lagrange equations. This provides a connection with the calculus of variations.

Consider the functional associated to the equilibrium problem

$$I[w] = \int_0^1 \Psi \left(w_R, \frac{w}{R}, \frac{w}{R}; R \right) dR.$$

We ask for which integrands $\Psi(v_1, v_2, v_3; R) : \mathbb{R}^4 \rightarrow \mathbb{R}$ the functional I admits zero variational derivatives, $\frac{\delta I}{\delta w} = 0$; such integrands are called *null Lagrangians* and they satisfy the Euler-Lagrange equation

$$-\partial_R(\Psi_{,1}) + R^{-1}(\Psi_{,2} + \Psi_{,3}) = 0 \quad \text{for all functions } w(R). \quad (3.37)$$

If $w = w(R, t)$ also depends on time, the evolution of a null Lagrangian Ψ is described by

$$\partial_t \Psi = \partial_R(\Psi_{,1} \partial_t w) \quad (3.38)$$

where Ψ and $\Psi_{,i}$ are evaluated at $(w_R, \frac{w}{R}, \frac{w}{R}, R)$.

It is easily verified that $\Psi(v_1, v_2, v_3; R)$ selected by

$$v_1, \quad v_1 v_2 R, \quad v_1 v_3 R, \quad \text{or} \quad v_1 v_2 v_3 R^2$$

are null-Lagrangians. Applying (3.37) to Ω^i , $i = 1, 5, 6, 7$, defined by (3.29) we get

$$-\partial_R(\Omega_{,1}^i(\Gamma)) + R^{-1}(\Omega_{,2}^i(\Gamma) + \Omega_{,3}^i(\Gamma)) = 0, \quad i = 1, 5, 6, 7, \quad (3.39)$$

with $\Gamma = (w_R, \frac{w}{R}, \frac{w}{R})$ defined by (3.32).

3.2.2 A Symmetrizable Extension

The null-Lagrangian structure is used in [15] to embed the equations of three-dimensional elastodynamics to a hyperbolic system endowed with a convex entropy, and to construct a variational approximation scheme for the problem. We follow this procedure in order to achieve an augmented system for radial elastodynamics.

The evolution in time of

$$\Omega(\Gamma) = \left(w_R, \frac{w}{R}, \frac{w}{R}, \frac{w^2}{R}, w_R w, w_R w, w_R w^2 \right) \quad (3.40)$$

gives

$$\begin{aligned} \partial_t \Omega^1(\Gamma) &= \partial_t (w_R) = \partial_R v = \partial_R (\Omega_{,1}^1(\Gamma) v) \\ \partial_t \Omega^i(\Gamma) &= \partial_t \left(\frac{w}{R} \right) = v/R = R^{-1} (\Omega_{,2}^i(\Gamma) + \Omega_{,3}^i(\Gamma)) v \quad \text{for } i = 2, 3 \\ \partial_t \Omega^4(\Gamma) &= \partial_t \left(\frac{w^2}{R} \right) = 2 \left(\frac{w}{R} \right) v = R^{-1} (\Omega_{,2}^4(\Gamma) + \Omega_{,3}^4(\Gamma)) v \\ \partial_t \Omega^i(\Gamma) &= \partial_t (w_R w) = \partial_R (w v) = \partial_R (\Omega_{,1}^i(\Gamma) v) \quad \text{for } i = 5, 6 \\ \partial_t \Omega^7(\Gamma) &= \partial_t (w_R w^2) = \partial_R (w^2 v) = \partial_R (\Omega_{,1}^7(\Gamma) v). \end{aligned} \quad (3.41)$$

Note that (3.41)_{1,5,6,7} are precisely the equations (3.38) describing the evolution of null Lagrangians. By contrast, (3.41)_{2,3,4} describe the evolution of lower-order terms and do not have the structure of (3.38).

Equations (3.41) and (3.36) motivate an extension of radial elasticity:

$$\left\{ \begin{array}{l} R^2 \partial_t v = \partial_R \left(R^2 \frac{\partial G}{\partial \xi_i}(\Xi) \frac{\partial \Omega^i}{\partial v_1}(\Gamma) \right) - R \frac{\partial G}{\partial \xi_i}(\Xi) \left(\frac{\partial \Omega^i}{\partial v_2}(\Gamma) + \frac{\partial \Omega^i}{\partial v_3}(\xi) \right) \\ \partial_t w = v \\ \partial_t \xi_i = \partial_R (\Omega_{,1}^i(\Gamma) v), \quad i = 1, 5, 6, 7 \\ \partial_t \xi_i = R^{-1} (\Omega_{,2}^i(\Gamma) + \Omega_{,3}^i(\Gamma)) v, \quad i = 2, 3, 4 \end{array} \right.$$

where $\Gamma = (w_R, \frac{w}{R}, \frac{w}{R})$, subject to the constraints

$$\xi_2, \xi_3 > 0, \quad \xi_7 > 0, \quad (R, t) \in (0, 1) \times [0, \infty), \quad (3.42)$$

and the boundary conditions $w(1, t) = \xi_2(1, t) = \xi_3(1, t) = \lambda$. System (3.2.2) describes the evolution of the vector (v, w, Ξ) , and is provided with initial data (v_0, w_0, Ξ_0) .

The extension has the following properties:

- (a) If $\Xi(\cdot, 0) = \Omega(\Gamma^0)$ where $\Gamma^0 = (w'_0, \frac{w_0}{R}, \frac{w_0}{R})$, then $\Xi(R, t) = \Omega(\Gamma(R, t))$, where $\Gamma = (w_R, \frac{w}{R}, \frac{w}{R})$. In other words, radial elasticity (3.33) can be viewed as a constrained evolution of (3.2.2).
- (b) The enlarged system admits an entropy pair

$$\partial_t \left(R^2 \left(\frac{v^2}{2} + G(\Xi) \right) \right) - \partial_R \left(R^2 \frac{\partial G}{\partial \xi_i}(\Xi) \frac{\partial \Omega^i}{\partial v_1}(Z) v \right) = 0, \quad (3.43)$$

with strictly convex entropy

$$\eta(v, \Xi) = \frac{v^2}{2} + G(\Xi). \quad (3.44)$$

The identity (3.43) holds for general solutions (v, w, Ξ) of (3.2.2) and is derived upon using the property (3.39) for the null Lagrangians (3.29).

3.2.3 An Alternative Extension with a Convex Entropy

System (3.2.2) provides an extension of radial elasticity that is endowed with a convex entropy. Concerning the objective of achieving a variational approximation, it has the drawback that the constraint (3.42) of positivity for the variables ξ_2, ξ_3 and ξ_7 is not preserved at the level of time-step approximations. Although one can control the positivity of ξ_7 (the augmented variable standing for the determinant), it is not possible to control the positivity of ξ_2, ξ_3 . There are also difficulties in proving that minimizers satisfy the corresponding Euler-Lagrange equations, the time-discretized system associated to (3.2.2).

For this reason, we develop an alternative extension by combining the evolution of null-Lagrangians with a change of variables used in Ball [3] for the equilibrium problem. This extension induces a variational approximation scheme that preserves the positivity of determinants.

The stored energy Φ is expressed in the form

$$\begin{aligned}\Phi(v_1, v_2, v_3) &= \bar{G}(v_1, v_2, v_3, v_2 v_3, v_1 v_3, v_1 v_2, v_1 v_2 v_3) \\ &= G(\Omega(V; \rho); \rho)\end{aligned}\tag{3.45}$$

where Ω and G are nonhomogeneous functions of ρ that are redefined so that

$$\Omega(V; \rho) := (v_1, v_2^3, v_3^3, v_2 v_3 \rho^{1/3}, v_1 v_3 \rho^{1/3}, v_1 v_2 \rho^{1/3}, v_1 v_2 v_3 \rho^{2/3})\tag{3.46}$$

$$G(\Xi; \rho) := \bar{G}(\xi_1, \xi_2^{1/3}, \xi_3^{1/3}, \xi_4/\rho^{1/3}, \xi_5/\rho^{1/3}, \xi_6/\rho^{1/3}, \xi_7/\rho^{2/3}).\tag{3.47}$$

It is now assumed that $G(\Xi; \rho)$ is a convex function of Ξ ; this is a somewhat stronger hypothesis than polyconvexity (which is convexity of \bar{G}) because of the definition of $\Omega^i(V; \rho)$, $i = 2, 3$, in (3.46). In the sequel any explicit ρ -dependence will be suppressed.

A change of variables. Following [3] we perform the change of variables

$$\rho = R^3 \quad \text{and} \quad \alpha = w^3.\tag{3.48}$$

Then $\Gamma = (w_R, \frac{w}{R}, \frac{w}{R})$ is expressed as

$$\Gamma = (\alpha_\rho(\rho/\alpha)^{2/3}, (\alpha/\rho)^{1/3}, (\alpha/\rho)^{1/3})\tag{3.49}$$

and the stored energy reads

$$\begin{aligned}W(\nabla y) &= \Phi(\alpha_\rho(\rho/\alpha)^{2/3}, (\alpha/\rho)^{1/3}, (\alpha/\rho)^{1/3}) \\ &= G(\Omega(\Gamma; \rho); \rho)\end{aligned}\tag{3.50}$$

where Ω and G are defined in (3.46), (3.47), and $G(\cdot; \rho)$ is convex.

The system (3.33) takes the form

$$\left\{ \begin{array}{l} \partial_t v = \partial_\rho \left(3\rho^{2/3} \frac{\partial G}{\partial \xi_i}(\Omega(\Gamma)) \frac{\partial \Omega^i}{\partial v_1}(\Gamma) \right) \\ \quad - \rho^{-1/3} \frac{\partial G}{\partial \xi_i}(\Omega(\Gamma)) \left(\frac{\partial \Omega^i}{\partial v_2}(\Gamma) + \frac{\partial \Omega^i}{\partial v_3}(\Gamma) \right) \\ \partial_t(\alpha^{1/3}) = v \\ \alpha(1) = \lambda, \alpha \geq 0, \alpha_\rho > 0, \quad (R, t) \in (0, 1) \times [0, \infty). \end{array} \right. \quad (3.51)$$

with the last inequalities encoding the constraints for solutions to represent elastic motions. In the new variables, by (3.46),

$$\Omega(\Gamma) = \left(\frac{\alpha_\rho}{\alpha^{2/3}} \rho^{2/3}, \frac{\alpha}{\rho}, \frac{\alpha}{\rho}, \frac{\alpha^{2/3}}{\rho^{1/3}}, \frac{\alpha_\rho}{\alpha^{1/3}} \rho^{2/3}, \frac{\alpha_\rho}{\alpha^{1/3}} \rho^{2/3}, \alpha_\rho \rho^{2/3} \right) \quad (3.52)$$

and, using (3.51)₂, we compute

$$\begin{aligned} \partial_t \Omega^1(\Gamma) &= \partial_t (3\rho^{2/3} \partial_\rho(\alpha^{1/3})) = 3\rho^{2/3} \partial_\rho v \\ \partial_t \Omega^i(\Gamma) &= \partial_t (\alpha/\rho) = 3\alpha^{2/3} v / \rho \quad i = 2, 3 \\ \partial_t \Omega^4(\Gamma) &= \partial_t (\alpha^{2/3} / \rho^{1/3}) = 2\alpha^{1/3} v / \rho^{1/3} \\ \partial_t \Omega^i(\Gamma) &= \partial_t ((3/2)\rho^{2/3} \partial_\rho(\alpha^{2/3})) = 3\rho^{2/3} \partial_\rho (\alpha^{1/3} v) \quad i = 5, 6 \\ \partial_t \Omega^7(\Gamma) &= \partial_t (\alpha_\rho \rho^{2/3}) = 3\rho^{2/3} \partial_\rho (\alpha^{2/3} v). \end{aligned} \quad (3.53)$$

These identities are summarized in two groups

$$\begin{aligned} \partial_t \Omega^i(\Gamma) &= 3\rho^{2/3} \partial_\rho (\Omega_{,1}^i(\Gamma) v), \quad i = 1, 5, 6, 7, \\ \partial_t \Omega^i(\Gamma) &= \rho^{-1/3} (\Omega_{,2}^i(\Gamma) + \Omega_{,3}^i(\Gamma)) v, \quad i = 2, 3, 4, \end{aligned} \quad (3.54)$$

the former representing the evolution of null-Lagrangians and the latter the evolution of lower order terms. The identities (3.39) satisfied by null-Lagrangians become

$$-3\rho^{2/3} \partial_\rho (\Omega_{,1}^i(\Gamma)) + \rho^{-1/3} (\Omega_{,2}^i(\Gamma) + \Omega_{,3}^i(\Gamma)) = 0, \quad i = 1, 5, 6, 7. \quad (3.55)$$

The augmented system. Next, consider the augmented system

$$\left\{ \begin{array}{l} \partial_t v = \partial_\rho (3\rho^{2/3} G_{,i}(\Xi) \Omega_{,1}^i(\Gamma)) - \rho^{-1/3} G_{,i}(\Xi) (\Omega_{,2}^i(\Gamma) + \Omega_{,3}^i(\Gamma)) \\ \partial_t \alpha^{1/3} = v \\ \partial_t \xi_i = 3\rho^{2/3} \partial_\rho (\Omega_{,1}^i(\Gamma) v), \quad i = 1, 5, 6, 7 \\ \partial_t \xi_i = \rho^{-1/3} (\Omega_{,2}^i(\Gamma) + \Omega_{,3}^i(\Gamma)) v, \quad i = 2, 3, 4 \end{array} \right. \quad (3.56)$$

where Γ is given by (3.49), subject to the boundary conditions and constraints, respectively,

$$\alpha(1) = \lambda, \alpha \geq 0, \alpha_\rho > 0, \quad (\rho, t) \in (0, 1) \times [0, \infty). \quad (3.57)$$

The system (3.56)₁-(3.56)₄ is a second-order system describing the evolution of the vector (v, α, Ξ) and is assigned initial data (v_0, α_0, Ξ_0) . It has the following properties:

(a) If $\Xi(\cdot, 0) = \Omega(\Gamma^0)$ with $\Gamma^0 = (\alpha'_0(\rho/\alpha_0)^{2/3}, (\alpha_0/\rho)^{1/3}, (\alpha_0/\rho)^{1/3})$, then $\Xi = \Omega(\Gamma)$ for all times. In other words, radial elasticity (3.33) can be viewed as a constrained evolution of (3.56).

(b) The enlarged system admits an entropy pair

$$\partial_t \left(\frac{v^2}{2} + G(\Xi) \right) - \partial_\rho (3\rho^{2/3} G_{,i}(\Xi) \Omega_{,1}^i(\Gamma) v) = 0 \quad (3.58)$$

with (for convex G) strictly convex entropy $\eta(v, \Xi) = \frac{v^2}{2} + G(\Xi)$.

At this point we set

$$\begin{aligned} \beta &= \alpha_\rho / \alpha^{2/3}, \quad \gamma = \alpha^{2/3}, \\ \Xi &= \left(\beta \rho^{2/3}, \frac{\alpha}{\rho}, \frac{\alpha}{\rho}, \frac{\gamma}{\rho^{1/3}}, \frac{3\gamma_\rho}{2} \rho^{2/3}, \frac{3\gamma_\rho}{2} \rho^{2/3}, \alpha_\rho \rho^{2/3} \right), \end{aligned} \quad (3.59)$$

and proceed to simplify the extended system working with α, β, γ, v as the independent variables.

Taking a closer look at the extended system we see that $\xi_2 = \xi_3$ by construction and hence equations (3.56)₂, $i = 2, 3$ are identical. Moreover,

$$\begin{aligned} \partial_t \xi_2 = 3\alpha^{2/3}v/\rho &\Rightarrow \partial_t \xi_7 = \rho^{2/3}\partial_\rho(\rho \partial_t \xi_2), \\ \partial_t \xi_4 = 2\alpha^{1/3}v/\rho^{1/3} &\Rightarrow \partial_t \xi_5 = \partial_t \xi_6 = \frac{3}{2}\rho^{2/3}\partial_\rho(\rho^{1/3}\partial_t \xi_4). \end{aligned} \quad (3.60)$$

Hence (3.56) is overdetermined and extra equations (3.56)₃ for $i = 5, 6, 7$ and (3.56)₃ for $i = 3$ can be excluded. In explicit form the extension (3.56) is written as

$$\left\{ \begin{array}{l} \partial_t v = \partial_\rho (3\rho^{2/3} G_{,i}(\Xi) \Omega_{,1}^i(\Gamma)) - \rho^{-1/3} G_{,i}(\Xi) (\Omega_{,2}^i(\Gamma) + \Omega_{,3}^i(\Gamma)) \\ \partial_t \beta = \partial_\rho(3v) \\ \partial_t \alpha = 3\alpha^{2/3}v \\ \partial_t \gamma = 2\alpha^{1/3}v \\ \alpha(1) = \lambda, \alpha \geq 0, \alpha_\rho > 0, (\rho, t) \in (0, 1) \times [0, \infty), \end{array} \right. \quad (3.61)$$

where from (3.61)₃ and (3.61)₄ we can derive the excluded equations

$$\begin{aligned} \partial_t \alpha_\rho &= \partial_\rho(3\alpha^{2/3}v) \\ \partial_t \gamma_\rho &= \partial_\rho(2\alpha^{1/3}v). \end{aligned} \quad (3.62)$$

3.3 Variational Approximation Scheme

The purpose of this section is to introduce a variational approximation scheme for the radial equation of elastodynamics with polyconvex and isotropic stored energy. The method we present here is inspired by the variational approximation

scheme for three dimensional elastodynamics proposed in [15] and described in Chapter 2. However, there are very important differences between the two schemes. The results of [15] do not take into account the constraint of positive determinant, $\det \nabla y > 0$, necessary to interpret y as a physically realizable motion. In this work, we consider the equations of radial elasticity (3.16) and proceed to devise a variational scheme that on one hand preserves the positivity of determinants expressed by (3.15) and on the other produces a time-discretized variant of entropy dissipation. Due to the lack of convexity of the stored energy W , similar to [15], we develop method based on the time-discretization of the extended system (3.56), which in the explicit form expressed by (3.61), and equipped with the convex entropy (3.58).

3.3.1 Time-discretized System

The general approach is to discretize the extended system (3.56) by use of implicit-explicit scheme. Successive iterates are constructed by discretizing (3.56) as follows: Given the $(j-1)^{th}$ iterate $(\alpha_0, \beta_0, \gamma_0, v_0)$ with $\alpha_0(\rho) \geq 0$ and $\alpha'_0(\rho) > 0$, $\rho \in (0, 1)$, we define $\Xi^0 = (\xi_i^0)_{i=1}^7$ by

$$\Xi^0(\rho) = \left(\beta_0 \rho^{2/3}, \frac{\alpha_0}{\rho}, \frac{\alpha_0}{\rho}, \frac{\gamma_0}{\rho^{1/3}}, \frac{3\gamma'_0}{2} \rho^{2/3}, \frac{3\gamma'_0}{2} \rho^{2/3}, \alpha'_0 \rho^{2/3} \right) \quad (3.63)$$

and construct the j^{th} iterate $(\alpha, \beta, \gamma, v)$, with corresponding $\Xi = (\xi_i)_{i=1}^7$ defined by

$$\Xi(\rho) = \left(\beta \rho^{2/3}, \frac{\alpha}{\rho}, \frac{\alpha}{\rho}, \frac{\gamma}{\rho^{1/3}}, \frac{3\gamma'}{2} \rho^{2/3}, \frac{3\gamma'}{2} \rho^{2/3}, \alpha' \rho^{2/3} \right), \quad (3.64)$$

by solving

$$\left\{ \begin{array}{l} (v - v_0)/h = \frac{d}{d\rho} (3\rho^{2/3} G_{,i}(\Xi) \Omega_{,1}^i(\Gamma^0)) \\ \quad - \rho^{-1/3} G_{,i}(\Xi) (\Omega_{,2}^i(\Gamma^0) + \Omega_{,3}^i(\Gamma^0)) \\ (\xi_i - \xi_i^0)/h = 3\rho^{2/3} \frac{d}{d\rho} (\Omega_{,1}^i(\Gamma^0) v), \quad i = 1, 5, 6, 7 \\ (\xi_i - \xi_i^0)/h = \rho^{-1/3} (\Omega_{,2}^i(\Gamma^0) + \Omega_{,3}^i(\Gamma^0)) v, \quad i = 2, 3, 4 \\ \xi_2(1) = \xi_3(1) = \lambda, \quad \xi_2, \xi_3 \geq 0, \quad \xi_7 > 0, \quad \rho \in (0, 1), \end{array} \right. \quad (3.65)$$

where

$$\Gamma = (\alpha'(\rho/\alpha)^{2/3}, (\alpha/\rho)^{1/3}, (\alpha/\rho)^{1/3}), \quad (3.66)$$

$$\Gamma^0 = (\alpha'_0(\rho/\alpha_0)^{2/3}, (\alpha_0/\rho)^{1/3}, (\alpha_0/\rho)^{1/3}). \quad (3.67)$$

As in the continuous case the discrete system (3.65) is overdetermined with extra equations

$$\begin{aligned} (\alpha_\rho - \alpha_{0\rho})/h &= \frac{d}{d\rho} (3\alpha_0^{2/3} v), \\ (\gamma_\rho - \gamma_{0\rho})/h &= \frac{d}{d\rho} (2\alpha_0^{1/3} v), \end{aligned} \quad (3.68)$$

corresponding to (3.65)₂, $i = 5, 6, 7$. Excluding them from the system above we get

$$\left\{ \begin{array}{l} (v - v_0)/h = \frac{d}{d\rho} (3\rho^{2/3} G_{,i}(\Xi) \Omega_{,1}^i(\Gamma^0)) \\ \quad - \rho^{-1/3} G_{,i}(\Xi) (\Omega_{,2}^i(\Gamma^0) + \Omega_{,3}^i(\Gamma^0)) \\ (\beta - \beta_0)/h = \frac{d}{d\rho} (3v) \\ (\alpha - \alpha_0)/h = 3\alpha_0^{2/3} v \\ (\gamma - \gamma_0)/h = 2\alpha_0^{1/3} v \\ \alpha(1) = \lambda, \quad \alpha \geq 0, \quad \alpha' > 0, \quad \rho \in (0, 1). \end{array} \right. \quad (3.69)$$

Note that equations (3.68) can be derived from (3.69)_{3,4}.

3.3.2 Discrete Entropy Inequality

Time-step approximations capture a subtle form of dissipation associated with the underlying variational structure and the convexity of the entropy, [14, 15]. Indeed, solutions of (3.69) satisfy a discrete entropy inequality: To see that, consider a *smooth* solution (Ξ, v) to (3.65) associated to smooth initial data (Ξ^0, v_0) given by (3.63). Multiplying (3.65)₁ by v we get

$$\begin{aligned} \frac{v(v - v_0)}{h} + G_{,i}(\Xi) \left(3\rho^{2/3} \Omega_{,1}^i(\Gamma^0) \frac{dv}{d\rho} + \rho^{-1/3} (\Omega_{,2}^i(\Gamma^0) + \Omega_{,3}^i(\Gamma^0)) v \right) \\ = \frac{d}{d\rho} (3\rho^{2/3} G_{,i}(\Xi) \Omega_{,1}^i(\Gamma^0) v). \end{aligned} \quad (3.70)$$

Then denoting

$$A_i = 3\rho^{2/3} \Omega_{,1}^i(\Gamma^0) \frac{dv}{d\rho} + \rho^{-1/3} (\Omega_{,2}^i(\Gamma^0) + \Omega_{,3}^i(\Gamma^0)) v, \quad i = 1, \dots, 7 \quad (3.71)$$

we claim

$$A_i = \frac{\xi_i - \xi_i^0}{h}. \quad (3.72)$$

Indeed, for $i = 2, 3, 4$ we have $\Omega_{,1}^i = 0$ and hence (3.65)₃ and (3.71) imply (3.72).

For $i = 1, 5, 6, 7$ by the properties (3.55) of null Lagrangians and (3.65)₂ we get

$$\begin{aligned} A_i = v \left(-3\rho^{2/3} \frac{d}{d\rho} (\Omega_{,1}^i(\Gamma^0)) + \rho^{-1/3} (\Omega_{,2}^i(\Gamma^0) + \Omega_{,3}^i(\Gamma^0)) \right) \\ + 3\rho^{2/3} \frac{d}{d\rho} (\Omega_{,1}^i(\Gamma^0) v) = (\xi_i - \xi_i^0) / h, \quad i = 1, 5, 6, 7. \end{aligned} \quad (3.73)$$

Thus (3.70) and (3.72) imply

$$\frac{1}{h} (v(v - v_0) + G_{,i}(\Xi) (\xi_i - \xi_i^0)) = \frac{d}{d\rho} (3\rho^{2/3} G_{,i}(\Xi) \Omega_{,1}^i(\Gamma^0) v).$$

Now, we denote $\Theta = (v, \Xi)$ and $\Theta^0 = (v_0, \Xi^0)$. Then $\eta = 1/2 v^2 + G(\Xi)$ satisfies

$$\frac{1}{h} D\eta \cdot (\Theta - \Theta^0) - \frac{d}{d\rho} (3\rho^{2/3} G_{,i}(\Xi) \Omega_{,1}^i(\Gamma^0) v) = 0.$$

For G convex the following identity holds

$$\frac{\eta(\Theta) - \eta(\Theta^0)}{h} - \frac{d}{d\rho} (3\rho^{2/3} G_{,i}(\Xi) \Omega_{,1}^i(\Gamma^0) v) \leq 0. \quad (3.74)$$

Remark 3.2. To obtain the inequality (3.74) we assumed that both the initial data (v_0, Ξ^0) and solution (v, Ξ) to (3.65) are smooth. Later in the sequel (Section 3.3.5) it will be shown that the actual iterates produced by the scheme via minimization satisfy (3.65) a.e. $\rho \in (0, 1)$ and hence one can deduce the inequality (3.74) for a.e. $\rho \in (0, 1)$ following the above calculations.

Remark 3.3. We have not studied in this work the convergence as the time-step $h \rightarrow 0$. For the three-dimensional elasticity equations this process produces measure-valued solutions [15] while for one-dimensional elasticity it gives entropy weak solutions [14]. In the present case we would expect to obtain weak solutions, but the compactness properties of (3.33) are not at present sufficiently understood. There are two differences of (3.33) relative to the well understood compactness theory of one-dimensional elasticity: the dependence of the stress on lower order terms, and the singularity at $R = 0$. Nevertheless, if the iterates u^h, v^h converge strongly, the discrete entropy inequality (3.74) gives a weak solution (v, Ξ) dissipating the mechanical energy, i.e.

$$\partial_t \left(\frac{v^2}{2} + G(\Xi) \right) - \partial_\rho (3\rho^{2/3} G_{,i}(\Xi) \Omega_{,1}^i(\Gamma) v) \leq 0. \quad (3.75)$$

3.3.3 Assumptions on the Stored-Energy Function

Henceforth, we consider stored-energy functions (3.45) of the form

$$\begin{aligned}\Phi(v_1, v_2, v_3) &= \bar{G}(v_1, v_2, v_3, v_3, v_1 v_3, v_1 v_2, v_1 v_2 v_3) \\ &= \varphi(v_1) + \varphi(v_2) + \varphi(v_3) + g(v_2 v_3) + g(v_1 v_3) + g(v_1 v_2) + h(v_1 v_2 v_3).\end{aligned}\tag{3.76}$$

Then, the function G defined in (3.47) reads

$$\begin{aligned}G(\Xi; \rho) &= \varphi(\xi_1) + \varphi(\xi_2^{1/3}) + \varphi(\xi_3^{1/3}) \\ &\quad + g(\xi_4 \rho^{1/3}) + g(\xi_5 \rho^{1/3}) + g(\xi_6 \rho^{1/3}) + h(\xi_7 \rho^{2/3}).\end{aligned}\tag{3.77}$$

Now, define $\psi(x) = \varphi(x^{1/3})$. Then, with Ξ defined in (3.64), the above is expressed by

$$G(\Xi) = \varphi(\beta \rho^{2/3}) + 2\psi(\alpha/\rho) + g(\gamma/\rho^{2/3}) + 2g(3\gamma'\rho^{1/3}/2) + h(\alpha').\tag{3.78}$$

We place the following assumptions on the functions φ, ψ, g, h appearing above:

$$(A1) \quad \lim_{\delta \rightarrow 0+} h(\delta) = \lim_{\delta \rightarrow +\infty} h(\delta)/\delta = +\infty;$$

$$(A2) \quad \varphi, \psi, g \in C^2(\mathbb{R}) \text{ and } h \in C^2(\mathbb{R}_+) \text{ satisfy}$$

$$\varphi, \psi, g, h, \varphi'', \psi'', g'' \geq 0 \quad \text{and} \quad h'' > 0;\tag{3.79}$$

$$(A3) \quad \text{For } 1 < p, q < \infty \text{ and some constants } c_1, c_2 > 0$$

$$\lim_{x \rightarrow \infty} \frac{\varphi(x)}{|x|^{3p}} = \lim_{x \rightarrow \infty} \frac{\psi(x)}{|x|^p} = c_1, \quad \lim_{x \rightarrow \infty} \frac{g(x)}{|x|^q} = c_2;\tag{3.80}$$

$$(A4) \quad \text{For } 1 < p, q < \infty \text{ as in (A3) and } C_1, C_2, C_3 > 0$$

$$\limsup_{x \rightarrow \infty} \frac{|\varphi'(x)|}{|x|^{3p-1}} \leq C_1, \quad \limsup_{x \rightarrow \infty} \frac{|\psi'(x)|}{|x|^{p-1}} \leq C_2, \quad \limsup_{x \rightarrow \infty} \frac{|g'(x)|}{|x|^q} \leq C_3;\tag{3.81}$$

In particular, G is *convex*.

Finally, we define spaces of functions on the interval $\rho \in (0, 1)$

$$X_1 = \{f(\rho) \in W^{1,1}(0, 1) : f/\rho \in L^p(0, 1)\},$$

$$X_2 = \{f(\rho) \in L^1_{loc}(0, 1) : f\rho^{2/3} \in L^{3p}(0, 1)\},$$

$$X_3 = \{f(\rho) \in W^{1,1}_{loc}(0, 1) : f/\rho^{2/3} \in L^q, f'\rho^{1/3} \in L^q(0, 1)\},$$

$$Y = \{f(\rho) \in W^{1,1}_{loc}(0, 1) : f \in L^2, f'\rho^{2/3} \in L^{3p}(0, 1)\},$$

and

$$X = X_1 \otimes X_2 \otimes X_3 \otimes Y.$$

3.3.4 Minimization Problem

We fix a parameter $\lambda > 0$ and for the initial data $(\alpha_0, \beta_0, \gamma_0, v_0) \in X$ require

$$\begin{cases} \alpha_0(1) = \lambda, & \alpha_0 \geq 0, \quad \alpha'_0 > 0, \text{ a.e. } \rho \in (0, 1), \\ \int_0^1 \frac{1}{2}v_0^2 + G(\Xi^0) d\rho < \infty. \end{cases} \quad (3.82)$$

Consider the problem of minimizing the functional

$$\begin{aligned} I(\alpha, \beta, \gamma, v) &= \int_0^1 \frac{1}{2}(v - v_0)^2 + G(\Xi) d\rho \\ &= \int_0^1 \frac{1}{2}(v - v_0)^2 + \varphi(\beta\rho^{2/3}) + 2\psi(\alpha/\rho) \\ &\quad + g(\gamma/\rho^{2/3}) + 2g(3\gamma'\rho^{1/3}/2) + h(\alpha') d\rho \end{aligned} \quad (3.83)$$

over the admissible set

$$\mathcal{A}_\lambda = \{(\alpha, \beta, \gamma, v) \in X : \alpha(0) \geq 0, \alpha(1) = \lambda, \alpha' > 0 \text{ a.e. and}$$

$$\begin{aligned} I(\alpha, \beta, \gamma, v) < \infty, \frac{(\beta - \beta_0)}{h} &= 3v', \\ \frac{(\alpha - \alpha_0)}{h} &= 3\alpha_0^{2/3}v, \frac{(\gamma - \gamma_0)}{h} = 2\alpha_0^{1/3}v\}. \end{aligned} \quad (3.84)$$

Remark 3.4. The differential constraints in (3.84) are affine, the condition $\alpha(1) = \lambda$ corresponds to the imposed boundary condition $y(x, t) = \lambda x$, $x \in \partial\mathcal{B}$, while $\alpha' > 0$ secures the positivity of determinants (3.15). We note also that I is well-defined for $(\alpha, \beta, \gamma, v) \in X$ with $\alpha' > 0$ a.e. $\rho \in (0, 1)$, though it might be equal to ∞ .

Lemma 3.5. *The admissible set \mathcal{A}_λ is nonempty.*

Proof. Take $(\alpha, \beta, \gamma, v) = (\alpha_0, \beta_0, \gamma_0, 0) \in X$. Then (3.82) implies $\alpha(0) \geq 0$, $\alpha(1) = \lambda$, $\alpha' > 0$ a.e. and

$$I(\alpha, \beta, \gamma, v) = \int_0^1 \frac{1}{2}v_0^2 + G(\Xi^0) d\rho < \infty.$$

Moreover the following holds: $(\beta - \beta_0)/h = 0 = 3v'$, $(\alpha - \alpha_0)/h = 0 = 3\alpha_0^{2/3}v$, and $(\gamma - \gamma_0)/h = 0 = 2\alpha_0^{1/3}v$. Hence $(\alpha, \beta, \gamma, v) \in \mathcal{A}_\lambda$. \square

Lemma 3.6 (I-bounded sequences). *Let $\{(\alpha_n, \beta_n, \gamma_n, v_n)\}_{n \in \mathbb{N}} \subset \mathcal{A}_\lambda$ and*

$$M = \sup_{n \in \mathbb{N}} I(\alpha_n, \beta_n, \gamma_n, v_n) < \infty. \quad (3.85)$$

Then $\exists (\alpha, \beta, \gamma, v) \in X$ and a subsequence $\{(\alpha_\mu, \beta_\mu, \gamma_\mu, v_\mu)\}$ s.t.

$$\begin{aligned}
\alpha_\mu &\rightharpoonup \alpha && \text{in } W^{1,1}, && \alpha_\mu/\rho &\rightharpoonup \alpha/\rho && \text{in } L^p, \\
\gamma_\mu/\rho^{2/3} &\rightharpoonup \gamma/\rho^{2/3} && \text{in } L^q, && \gamma'_\mu \rho^{1/3} &\rightharpoonup \gamma' \rho^{1/3} && \text{in } L^q, \\
v_\mu &\rightharpoonup v && \text{in } L^2, && v'_\mu \rho^{2/3} &\rightharpoonup v' \rho^{2/3} && \text{in } L^{3p}, \\
\beta_\mu \rho^{2/3} &\rightharpoonup \beta \rho^{2/3} && \text{in } L^{3p}.
\end{aligned} \tag{3.86}$$

Proof. First, $\alpha_n \geq 0$, $\alpha'_n > 0$ a.e. and $\alpha_n(1) = \lambda$ imply that $|\alpha_n| \leq \lambda$. Second, from (3.85) it follows $\int_0^1 h(\alpha'_n) d\rho < M$, $\forall n$. By the de la Vallée Poussin criterion there exists $\alpha \in W^{1,1}$ and a subsequence $\{\alpha_s\}$ such that $\alpha_s \rightharpoonup \alpha$ weakly in $W^{1,1}$.

By (A3) there exist constants C_1, C_2 s.t. $\varphi(x) \geq C_1|x|^{3p} - C_2$, $\psi(x) \geq C_1|x|^p - C_2$ and $g(x) \geq C_1|x|^q - C_2$, and thus

$$\begin{aligned}
M &\geq I(\alpha_s, \beta_s, \gamma_s, v_s) \geq \int_0^1 \frac{1}{2}(v_s - v_0)^2 d\rho \\
&\quad + C_1 \int_0^1 |\beta_s \rho^{2/3}|^{3p} + 2|\alpha_s/\rho|^p + |\gamma_s/\rho^{2/3}|^q + \frac{3}{2}|\gamma'_s \rho^{1/3}|^q d\rho - 4C_2
\end{aligned} \tag{3.87}$$

This implies for $1 < p, q < \infty$ that $\alpha/\rho \in L^p$ and there exist $\beta \in X_2, \gamma \in X_3$, and $v \in L^2$ and a subsequence $\{\alpha_\mu, \beta_\mu, \gamma_\mu, v_\mu\}$ of $\{\alpha_s, \beta_s, \gamma_s, v_s\}$ such that (3.86)_{2,3,4,5,6} hold.

Finally, as $(\alpha_\mu, \beta_\mu, \gamma_\mu, v_\mu) \in A_\lambda$ we have $3v'_\mu \rho^{2/3} = (\beta_\mu - \beta_0) \rho^{2/3}/h$. Then by (3.86)₃ we get $3v'_\mu \rho^{2/3} \rightharpoonup (\beta - \beta_0) \rho^{2/3}/h$ in L^{3p} . Then by (3.86)₆ for each $f \in C_0^\infty(0, 1)$

$$\begin{aligned}
\int_0^1 v f' d\rho &= \lim_{\mu \rightarrow \infty} \int_0^1 v_\mu f' d\rho \\
&= - \lim_{\mu \rightarrow \infty} \int_0^1 v'_\mu f d\rho = - \int_0^1 \frac{1}{3h} (\beta - \beta_0) f d\rho
\end{aligned} \tag{3.88}$$

and hence $v' = (\beta - \beta_0)/3h$. Therefore $v \in Y$ and $v'_\mu \rho^{2/3} \rightharpoonup v' \rho^{2/3}$. \square

Theorem 3.2 (Lower semi-continuity). *Let $\{(\alpha_n, \beta_n, \gamma_n, v_n)\}_{n \in \mathbb{N}} \subset \mathcal{A}_\lambda$, $(\alpha, \beta, \gamma, v) \in X$ satisfy (3.85) and (3.86). Then $(\alpha, \beta, \gamma, v) \in \mathcal{A}_\lambda$ and*

$$I(\alpha, \beta, \gamma, v) \leq \liminf_{n \rightarrow \infty} I(\alpha_n, \beta_n, \gamma_n, v_n) = s < \infty. \quad (3.89)$$

Proof. By hypothesis $0 \leq I_n = I(\alpha_n, \beta_n, \gamma_n, v_n) \leq M$, $\forall n \in \mathbb{N}$ and thus $s < \infty$.

Recall that $\alpha_n \rightharpoonup \alpha$ weakly in $W^{1,1}$ and (along a subsequence) uniformly on $C[0, 1]$.

Since $\alpha_n(1) = \lambda$ we obtain $\alpha(1) = \lambda$. Moreover,

$$\lim_{n \rightarrow \infty} \int_0^1 \alpha'_n \chi_{\{\alpha' < 0\}} d\rho = \int_0^1 \alpha' \chi_{\{\alpha' < 0\}} d\rho. \quad (3.90)$$

Since $\alpha'_n > 0$ a.e. we obtain $\int_0^1 \alpha' \chi_{\{\alpha' < 0\}} d\rho \geq 0$, and thus $m\{\alpha' < 0\} = 0$.

Now, denote $A = \{\rho \in (0, 1) : \alpha' = 0\}$ and show that $m(A) = 0$. We will argue by contradiction. Assume that $m(A) = \varepsilon > 0$. Then (3.86) implies

$$\lim_{n \rightarrow \infty} \int_0^1 \alpha'_n \chi_A d\rho = \int_0^1 \alpha' \chi_A d\rho = 0. \quad (3.91)$$

Then, as $\alpha'_n > 0$ a.e., $\lim_{n \rightarrow \infty} \int_0^1 |\alpha'_n \chi_A| d\rho = 0$. Hence $\alpha'_n \chi_A \rightarrow 0$ in L^1 . We extract a subsequence $\{\alpha'_{n_k}\}$ such that $\alpha'_{n_k} \chi_A \rightarrow 0$ a.e. $\rho \in (0, 1)$. Now, by Egoroff's theorem there exists a measurable set $B \subset A$ such that $m(B) > \varepsilon/2$ and $\alpha'_{n_k} \rightarrow 0$ uniformly on B . Next, observe that

$$\int_0^1 h(\alpha'_{n_k}) d\rho \geq \int_B h(\alpha'_{n_k}) d\rho \geq m(B) \left(\inf_{\rho \in B} h(\alpha'_{n_k}) \right) =: m(B) \mu_{n_k}$$

Since $\mu_{n_k} \rightarrow \infty$ this contradicts (3.85). We conclude that $m(A) = 0$.

Next we prove $\alpha \geq 0$ a.e. $\rho \in (0, 1)$. Again (3.86)₁ implies

$$\lim_{n \rightarrow \infty} \int_0^1 \alpha_n \chi_{\{\alpha < 0\}} d\rho = \int_0^1 \alpha \chi_{\{\alpha < 0\}} d\rho \geq 0, \quad (3.92)$$

and thus $m\{\alpha < 0\} = 0$. This concludes that α satisfies all restrictions of membership in \mathcal{A}_λ .

Next, by (A2) we get

$$\begin{aligned}
\varphi(\beta_n \rho^{2/3}) &\geq \varphi(\beta \rho^{2/3}) + \varphi'(\beta \rho^{2/3})(\beta_n - \beta) \rho^{2/3}, \\
\psi(\alpha_n / \rho) &\geq \psi(\alpha / \rho) + \psi'(\alpha / \rho)(\alpha_n - \alpha) / \rho, \\
g(\gamma_n / \rho^{2/3}) &\geq g(\gamma / \rho^{2/3}) + g'(\gamma / \rho^{2/3})(\gamma_n - \gamma) / \rho^{2/3}, \\
g(3\gamma'_n \rho^{1/3} / 2) &\geq g(3\gamma' \rho^{1/3} / 2) + g'(3\gamma' \rho^{1/3} / 2)(\gamma'_n - \gamma) 3\rho^{1/3} / 2
\end{aligned} \tag{3.93}$$

a.e. $\rho \in (0, 1)$. As $(\alpha, \beta, \gamma, v), (\alpha_n, \beta_n, \gamma_n, v_n) \in X$, from (A3) it follows that the right-hand side of each of the inequalities in (3.93) are integrable and

$$\begin{aligned}
\varphi'(\beta \rho^{2/3}) &\in L^{\frac{3p}{3p-1}}, \quad \psi'(\alpha / \rho) \in L^{\frac{p}{p-1}}, \\
\text{and } g'(\gamma / \rho^{2/3}), g'(3\gamma'_n \rho^{1/3} / 2) &\in L^{\frac{q}{q-1}}.
\end{aligned} \tag{3.94}$$

Take an arbitrary $0 < \delta < 1$ and set $A_\delta = \{\rho \in (0, 1) : \delta \leq \alpha' \leq 1/\delta\}$. Then by (A2)

$$h(\alpha'_n) \geq h(\alpha')\chi_{A_\delta} + h'(\alpha')(\alpha'_n - \alpha')\chi_{A_\delta} \quad \text{a.e. } \rho \in (0, 1). \tag{3.95}$$

Moreover, (A1) and (A2) together imply

$$\begin{aligned}
0 &\leq h(\alpha')\chi_{A_\delta} + |h'(\alpha')|\chi_{A_\delta} \\
&\leq 2 \max(h(\delta), h(1/\delta), |h'(\delta)|, |h'(1/\delta)|).
\end{aligned}$$

Hence

$$h(\alpha')\chi_{A_\delta}, h'(\alpha')\chi_{A_\delta} \in L^\infty, \tag{3.96}$$

and we conclude that the right-hand side of (3.95) is integrable.

Finally,

$$(v_n - v_0)^2 \geq (v - v_0)^2 + 2(v - v_0)(v_n - v) \quad \text{a.e. } \rho \in (0, 1), \quad (3.97)$$

where the right-hand side is integrable as $v, v_n, v_0 \in L^2$.

Following the discussion above, (3.93)-(3.97) imply

$$\begin{aligned} I_n &\geq \int_0^1 \frac{1}{2} (v - v_0)^2 + \varphi(\beta \rho^{2/3}) + 2\psi(\alpha/\rho) \\ &\quad + g(\gamma/\rho^{2/3}) + 2g(3\gamma'\rho^{1/3}/2) d\rho + \int_0^1 h(\alpha') \chi_{A_\delta} d\rho \\ &\quad + \int_0^1 (v - v_0)(v_n - v) + \varphi'(\beta \rho^{2/3})(\beta_n - \beta) \rho^{2/3} \\ &\quad + 2\psi'(\alpha/\rho)(\alpha_n - \alpha)/\rho + g'(\gamma/\rho^{2/3})(\gamma_n - \gamma)/\rho^{2/3} \\ &\quad + g'(3\gamma'\rho^{1/3}/2)(\gamma'_n - \gamma)3\rho^{1/3} + h'(\alpha') \chi_{A_\delta}(\alpha'_n - \alpha') d\rho \\ &= J + J_\delta + J_n. \end{aligned}$$

Then, letting $n \rightarrow \infty$, we obtain

$$\infty > s = \liminf_{n \rightarrow \infty} I_n \geq J + J_\delta + \liminf_{n \rightarrow \infty} J_n.$$

Now from (3.86), (3.94), (3.96), and $v - v_0 \in L^2$ it follows that $\lim_{n \rightarrow \infty} J_n = 0$ and hence

$$\infty > s = \liminf_{n \rightarrow \infty} I_n \geq J + \int_0^1 h(\alpha') \chi_{A_\delta} d\rho. \quad (3.98)$$

Now, as $\alpha' > 0$ a.e. $\rho \in (0, 1)$ and $\alpha' \in L^1$, the set $\{\alpha' = 0\} \cup \{\alpha' = \infty\}$ is of measure zero and hence

$$\lim_{\delta \rightarrow 0+} h(\alpha') \chi_{A_\delta} = h(\alpha') \chi_{\{0 < \alpha' < \infty\}} = h(\alpha') \quad \text{a.e. } \rho \in (0, 1). \quad (3.99)$$

Finally, let $\delta \rightarrow 0+$. Then from (3.98), (3.99) and Monotone Convergence Theorem it follows

$$\infty > s = \liminf_{n \rightarrow \infty} I_n \geq J + \int_0^1 h(\alpha') d\rho = I(\alpha, \beta, \gamma, v)$$

and hence (3.89) holds. Since $(\alpha_n, \beta_n, \gamma_n, v_n) \in \mathcal{A}_\lambda$, and the other constraints are linear, one easily checks that the limiting $(\alpha, \beta, \gamma, v) \in \mathcal{A}_\lambda$. \square

Theorem 3.3 (Existence). *There exists $(\alpha, \beta, \gamma, v) \in \mathcal{A}_\lambda$ satisfying*

$$I(\alpha, \beta, \gamma, v) = \inf_{\mathcal{A}_\lambda} I(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{v}). \quad (3.100)$$

Proof. As \mathcal{A}_λ is nonempty, we can set $s = \inf_{\mathcal{A}_\lambda} I(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{v})$. Then by definition of \mathcal{A}_λ we have $I(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{v}) < \infty$ for each $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{v}) \in \mathcal{A}_\lambda$. This implies that s is finite.

Next, by definition of s there exists $\{(\alpha_n, \beta_n, \gamma_n, v_n)\}_{n \in \mathbb{N}} \in \mathcal{A}_\lambda$ such that $\lim_{n \rightarrow \infty} I_n = s$ with $I_n = I(\alpha_n, \beta_n, \gamma_n, v_n)$. Then, as $\{I_n\}_{n \in \mathbb{N}}$ is bounded, lemma 3.6 and Theorem 3.2 imply that $\exists(\alpha, \beta, \gamma, v) \in \mathcal{A}_\lambda$ satisfying $I(\alpha, \beta, \gamma, v) \leq \liminf_{n \rightarrow \infty} I_n = s$. In this case the definition of s implies $I(\alpha, \beta, \gamma, v) = s$. \square

Theorem 3.4 (Uniqueness). *The minimizer $(\alpha, \beta, \gamma, v) \in \mathcal{A}_\lambda$ of I over \mathcal{A}_λ is unique.*

Proof. We will argue by contradiction. Assume $(\alpha, \beta, \gamma, v), (\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{v}) \in \mathcal{A}_\lambda$ are two distinct minimizers. Then we consider $(\frac{\alpha+\bar{\alpha}}{2}, \frac{\beta+\bar{\beta}}{2}, \frac{\gamma+\bar{\gamma}}{2}, \frac{v+\bar{v}}{2})$ and notice that it also belongs to \mathcal{A}_λ .

Define $A = \{\rho \in (0, 1) : \alpha' \neq \bar{\alpha}'\}$. Then $mA > 0$. Indeed, if $\alpha' = \bar{\alpha}'$ a.e., then $\alpha(1) = \bar{\alpha}(1) = \lambda$ implies $\alpha = \bar{\alpha}$. In turn, this implies $v = \bar{v}$, $\beta = \bar{\beta}$ and $\gamma = \bar{\gamma}$, which contradicts to the assumption that $(\alpha, \beta, \gamma, v)$ and $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{v})$ are distinct.

Now, as $h'' > 0$, we have

$$\frac{h(\alpha') + h(\bar{\alpha}')}{2} > h\left(\frac{\alpha' + \bar{\alpha}'}{2}\right), \quad \rho \in A,$$

and thus, as mA is positive,

$$\int_0^1 \frac{h(\alpha') + h(\bar{\alpha}')}{2} d\rho > \int_0^1 h\left(\frac{\alpha' + \bar{\alpha}'}{2}\right) d\rho.$$

Let $s = \inf_{\mathcal{A}_\lambda} I(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{v})$. Then by the inequality above and convexity of φ, ψ and g we obtain

$$s = \frac{I(\alpha, \beta, \gamma, v) + I(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{v})}{2} > I\left(\frac{\alpha + \bar{\alpha}}{2}, \frac{\beta + \bar{\beta}}{2}, \frac{\gamma + \bar{\gamma}}{2}, \frac{v + \bar{v}}{2}\right), \quad (3.101)$$

which, since $\left(\frac{\alpha + \bar{\alpha}}{2}, \frac{\beta + \bar{\beta}}{2}, \frac{\gamma + \bar{\gamma}}{2}, \frac{v + \bar{v}}{2}\right) \in \mathcal{A}_\lambda$, contradicts the definition of s . Hence $(\alpha, \beta, \gamma, v) = (\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{v})$. \square

3.3.5 Euler-Lagrange Equations

Next, we show that the minimizer of I satisfies the system (3.65) a.e. $\rho \in (0, 1)$.

To this end, in addition to (3.82), we assume that the initial iterate $(\alpha_0, \beta_0, \gamma_0, v_0)$ satisfies for each $\delta \in (0, 1)$

$$\alpha'_0 \in L^{3p}(\delta, 1) \bigcap L^q(\delta, 1). \quad (3.102)$$

Theorem 3.5 (Weak Form). *Let $(\alpha, \beta, \gamma, v) \in \mathcal{A}_\lambda$ be the minimizer of I over \mathcal{A}_λ and the initial iterate $(\alpha_0, \beta_0, \gamma_0, v_0)$ satisfy (3.82) and (3.102). Let also*

$$G_1(\rho) = G_{,i}(\Xi) \Omega_{,1}^i(\Gamma^0) \quad (3.103)$$

and

$$G_2(\rho) = G_{,i}(\Xi) (\Omega_{,2}^i(\Gamma^0) + \Omega_{,3}^i(\Gamma^0)) \quad (3.104)$$

Then, for each $\delta \in (0, 1)$,

$$\rho^{2/3}G_1(\rho) \in W^{1,1}(\delta, 1), \quad \rho^{-1/3}G_2(\rho) \in L^1(\delta, 1),$$

and for a.e. $\rho \in (0, 1)$

$$3\rho^{2/3}G_1(\rho) = \int_1^\rho \left(s^{-1/3}G_2(s) + \frac{v(s) - v_0(s)}{h} \right) ds + \text{const.} \quad (3.105)$$

Moreover, for each $\delta \in (0, 1)$,

$$\alpha' \in L^{3p}(\delta, 1) \bigcap L^q(\delta, 1). \quad (3.106)$$

Proof. Fix $k \in \mathbb{N}$ and define $S_k = \{\rho \in [1/k, 1) : 1/k < \alpha' < k\}$. Let $f \in L^\infty$ with $\int_{S_k} f d\rho = 0$. We denote by $\chi_k = \chi_{S_k}$, $l_k = \alpha_0(1/k)$ and set

$$\mu(\rho) = \int_0^\rho \chi_k(s) f(s) ds. \quad (3.107)$$

Before proceeding further we make the following remark. Let $t \in \mathbb{R}$ and $F(x) = x^t, x \in \mathbb{R}_+$. Take $\delta \in (0, 1)$. Then, as $\alpha_0 \in W^{1,1}$, $\alpha_0 \geq 0$ and $\alpha'_0 > 0$ a.e. $\rho \in (0, 1)$ we must have $0 < \alpha_0(\delta) \leq \alpha_0 \leq \lambda$ for all $\rho \in (\delta, 1)$. Hence $|F'(\alpha_0)| \leq t(\alpha_0(\delta) + \lambda)^{t-1}$ for all $\rho \in (\delta, 1)$. Therefore we conclude that for each $t \in \mathbb{R}$ and $\delta \in (0, 1)$

$$\alpha_0^t \in W^{1,1}(\delta, 1) \quad \text{with} \quad \frac{d}{d\rho} (\alpha_0^t) = t\alpha_0^{t-1}\alpha'_0. \quad (3.108)$$

(i) *Step 1. Definition of the variation.*

For $|\varepsilon| < \frac{1}{6k(\|f\|_\infty+1)}$ we define $(\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon, v_\varepsilon)$ by

$$\begin{aligned} v_\varepsilon &= v + \varepsilon \frac{\mu}{h\alpha_0^{2/3}} \\ \alpha_\varepsilon &= \alpha_0 + h(3v_\varepsilon\alpha_0^{2/3}) = \alpha + 3\varepsilon\mu \\ \beta_\varepsilon &= \beta_0 + h(3v'_\varepsilon) = \beta + 3\varepsilon \left(\frac{\mu}{\alpha_0^{2/3}} \right)' \\ \gamma_\varepsilon &= \gamma_0 + h(2v_\varepsilon\alpha_0^{1/3}) = \gamma + 2\varepsilon \frac{\mu}{\alpha_0^{1/3}}. \end{aligned} \tag{3.109}$$

Due to (3.108), $(\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon, v_\varepsilon)$ is well-defined. We next prove:

Lemma 3.7. *The variation $(\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon, v_\varepsilon) \in \mathcal{A}_\lambda$.*

Proof. First, we notice that

$$(\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon, v_\varepsilon) = (\alpha, \beta, \gamma, v) \quad \text{if } \rho \in (0, 1/k). \tag{3.110}$$

Then we check that

$$\alpha_\varepsilon(1) = \alpha(1) + 3\varepsilon \int_{S_k} f(s) ds = \lambda.$$

Next, we see that $\alpha'_\varepsilon = \alpha' + 3\varepsilon\chi_k f$ and therefore

$$\begin{aligned} \alpha'_\varepsilon &= \alpha', \quad \rho \notin S_k, \\ \frac{1}{2k} &\leq \alpha'_\varepsilon \leq k+1, \quad \rho \in S_k. \end{aligned} \tag{3.111}$$

This implies that $\alpha_\varepsilon > 0$ a.e. $\rho \in (0, 1)$ and hence (3.110) implies $\alpha_\varepsilon \geq 0$.

Now we make the following estimates. First, we see that

$$|\mu'| + \left| \frac{\mu}{\rho} \right| + \left| \frac{\mu}{\rho^{2/3}\alpha_0^{1/3}} \right| + \left| \frac{\mu}{h\alpha_0^{2/3}} \right| \leq \|f\|_\infty \left(1 + k + \frac{k^{2/3}}{l_k^{1/3}} + \frac{1}{hl_k^{2/3}} \right)$$

and for $j = 1, 2$

$$\left| \left(\frac{\mu}{\alpha_0^{j/3}} \right)' \right| \leq \|f\|_\infty \left(l_k^{-j/3} + l_k^{-(1+j/3)} |\alpha'_0| \right).$$

Thus we conclude that there exists C such that $\forall \rho \in (1/k, 1)$

$$|\alpha'_\varepsilon - \alpha'| + |\alpha_\varepsilon/\rho - \alpha/\rho| + |\gamma_\varepsilon/\rho^{2/3} - \gamma/\rho^{2/3}| + |v - v_\varepsilon| \leq \varepsilon C \quad (3.112)$$

and

$$|\beta_\varepsilon \rho^{2/3} - \beta \rho^{2/3}| + |\gamma'_\varepsilon \rho^{1/3} - \gamma' \rho^{1/3}| \leq \varepsilon C (1 + |\alpha'_0|). \quad (3.113)$$

As $(\alpha, \beta, \gamma, v) \in X$, the last two inequalities imply $(\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon, v_\varepsilon) \in X$.

Further, by (A3), (3.112) and (3.113) we conclude that there exists C such that for all $\rho \in (1/k, 1)$

$$\begin{aligned} \psi(\alpha_\varepsilon/\rho) &\leq C (|\alpha/\rho|^p + 1) \\ \varphi(\beta_\varepsilon \rho^{2/3}) &\leq C (|\beta \rho^{2/3}|^{3p} + |\alpha'_0|^{3p} + 1) \\ g(\gamma_\varepsilon/\rho^{2/3}) &\leq C (|\gamma/\rho^{2/3}|^q + 1) \\ g(3\gamma'_\varepsilon \rho^{1/3}/2) &\leq C (|\gamma' \rho^{1/3}|^q + |\alpha'_0|^q + 1). \end{aligned}$$

By (3.111) we also have

$$\begin{aligned} h(\alpha'_\varepsilon) &= h(\alpha'), \quad \rho \notin S_k, \\ h(\alpha'_\varepsilon) &\leq \max_{\frac{1}{2k} \leq \delta \leq k} |h(\delta)| = M_k, \quad \rho \in S_k \end{aligned} \quad (3.114)$$

and hence

$$h(\alpha'_\varepsilon) \leq h(\alpha') + M_k, \quad \rho \in (0, 1). \quad (3.115)$$

Now, similarly to (3.64), set

$$\Xi_\varepsilon = \left(\beta_\varepsilon \rho^{2/3}, \frac{\alpha_\varepsilon}{\rho}, \frac{\alpha_\varepsilon}{\rho}, \frac{\gamma_\varepsilon}{\rho^{1/3}}, \frac{3\gamma'_\varepsilon}{2} \rho^{2/3}, \frac{3\gamma'_\varepsilon}{2} \rho^{2/3}, \alpha'_\varepsilon \rho^{2/3} \right). \quad (3.116)$$

Then, by the discussion above, it follows that

$$G(\Xi_\varepsilon) + \frac{(v_\varepsilon - v_0)^2}{2} = G(\Xi) + \frac{(v - v_0)^2}{2}, \quad \rho \in (0, 1/k), \quad (3.117)$$

and there exists C such that for $\rho \in (1/k, 1)$

$$G(\Xi_\varepsilon) + \frac{(v_\varepsilon - v_0)^2}{2} \leq C \left(1 + |\beta \rho^{2/3}|^{3p} + |\alpha'_0|^{3p} + |\alpha/\rho|^p + |\gamma/\rho^{2/3}|^q \right. \\ \left. + |\gamma' \rho^{1/3}|^q + |\alpha'_0|^q + |v|^2 + |v_0|^2 + h(\alpha') \right). \quad (3.118)$$

As $I(\alpha, \beta, \gamma, v) < \infty$, (3.117) and (3.118) imply $I(\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon, v_\varepsilon) < \infty$ and hence by construction and the above discussion we get $(\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon, v_\varepsilon) \in \mathcal{A}_\lambda$. \square

Step 2. The next objective is to validate the formal identity

$$\frac{d}{d\varepsilon} I(\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon, v_\varepsilon) \Big|_{\varepsilon=0} = \int_0^1 \frac{d}{d\varepsilon} \left(\frac{(v_\varepsilon - v_0)^2}{2} + G(\Xi_\varepsilon) \right) \Big|_{\varepsilon=0} d\rho = 0. \quad (3.119)$$

This will require several detailed estimations presented below.

At this point, let us make estimates of the following difference quotients on the interval $\rho \in (1/k, 1)$. First, by (3.112) we get

$$\frac{1}{\varepsilon} |(v_\varepsilon - v_0)^2 - (v - v_0)^2| = \frac{1}{\varepsilon} |v_\varepsilon - v| |v_\varepsilon + v - 2v_0| \\ \leq C (|v| + |v_0| + 1). \quad (3.120)$$

Further, by the Mean Value Theorem

$$\frac{1}{\varepsilon} |\varphi(\beta_\varepsilon \rho^{2/3}) - \varphi(\beta \rho^{2/3})| = \frac{1}{\varepsilon} |\varphi'(\tau_\varepsilon)| |\beta_\varepsilon \rho^{2/3} - \beta \rho^{2/3}|,$$

where $\min(\beta, \beta_\varepsilon) \rho^{2/3} \leq \tau_\varepsilon \leq \max(\beta, \beta_\varepsilon) \rho^{2/3}$. Hence from (3.113) it follows $|\tau_\varepsilon| \leq |\beta \rho^{2/3}| + \varepsilon C (|\alpha'_0| + 1)$ and therefore (A4) implies

$$|\varphi'(\tau_\varepsilon)| \leq C (|\beta \rho^{2/3}|^{3p-1} + |\alpha'_0|^{3p-1} + 1).$$

Thus

$$\frac{1}{\varepsilon} |\varphi(\beta_\varepsilon \rho^{2/3}) - \varphi(\beta \rho^{2/3})| \\ \leq C (|\beta \rho^{2/3}|^{3p-1} + |\alpha'_0|^{3p-1} + 1) (|\alpha'_0| + 1). \quad (3.121)$$

Similarly,

$$\frac{1}{\varepsilon}|\psi(\alpha_\varepsilon/\rho) - \psi(\alpha/\rho)| = \frac{1}{\varepsilon}|\psi'(\tau_\varepsilon)||\alpha_\varepsilon/\rho - \alpha/\rho|,$$

where $\min(\alpha_\varepsilon, \alpha)/\rho \leq \tau_\varepsilon \leq \max(\alpha_\varepsilon, \alpha)/\rho$. Hence $|\tau_\varepsilon| \leq |\alpha/\rho| + \varepsilon C$ and (A4) implies

$$|\psi'(\tau_\varepsilon)| \leq C \left((|\alpha/\rho| + 1)^{p-1} + 1 \right)$$

and hence

$$\frac{1}{\varepsilon}|\psi(\alpha_\varepsilon/\rho) - \psi(\alpha/\rho)| \leq C \left((|\alpha/\rho| + 1)^{p-1} + 1 \right). \quad (3.122)$$

Next,

$$\frac{1}{\varepsilon}|g(\gamma_\varepsilon/\rho^{2/3}) - g(\gamma/\rho^{2/3})| = \frac{1}{\varepsilon}|g'(\tau_\varepsilon)||\gamma_\varepsilon/\rho^{2/3} - \gamma/\rho^{2/3}|,$$

where $\min(\gamma_\varepsilon, \gamma)/\rho^{2/3} \leq |\tau_\varepsilon| \leq \max(\gamma_\varepsilon, \gamma)/\rho^{2/3}$ and hence $|\tau_\varepsilon| \leq |\gamma/\rho^{2/3}| + \varepsilon C$. Then by (A4)

$$|g'(\tau_\varepsilon)| \leq C \left((|\gamma/\rho^{2/3}| + 1)^{q-1} + 1 \right)$$

and hence

$$\frac{1}{\varepsilon}|g(\gamma_\varepsilon/\rho^{2/3}) - g(\gamma/\rho^{2/3})| \leq C \left((|\gamma/\rho^{2/3}| + 1)^{q-1} + 1 \right). \quad (3.123)$$

Further,

$$\frac{1}{\varepsilon}|g(3\gamma'_\varepsilon\rho^{1/3}/2) - g(3\gamma'\rho^{1/3}/2)| = \frac{3}{2\varepsilon}|g'(\tau_\varepsilon)||\gamma'_\varepsilon\rho^{1/3} - \gamma'\rho^{1/3}|,$$

where $\frac{3}{2}\min(\gamma'_\varepsilon, \gamma')\rho^{1/3} \leq |\tau_\varepsilon| \leq \frac{3}{2}\max(\gamma'_\varepsilon, \gamma')\rho^{1/3}$. Hence we must have $|\tau_\varepsilon| \leq \frac{3}{2}(|\gamma'\rho^{1/3}| + \varepsilon C(|\alpha'_0| + 1))$. Then (A4) implies

$$|g'(\tau_\varepsilon)| \leq C \left((|\gamma'\rho^{1/3}| + |\alpha'_0| + 1)^{q-1} + 1 \right)$$

and hence

$$\begin{aligned} & \frac{1}{\varepsilon}|g(3\gamma'_\varepsilon\rho^{1/3}/2) - g(3\gamma'\rho^{1/3}/2)| \\ & \leq C \left((|\gamma'\rho^{1/3}| + |\alpha'_0| + 1)^{q-1} + 1 \right) (|\alpha'_0| + 1). \end{aligned} \quad (3.124)$$

Finally, if $\rho \notin S_k$, then $\frac{1}{\varepsilon}|h(\alpha'_\varepsilon) - h(\alpha')| = 0$ and if $\rho \in S_k$, we get

$$\frac{1}{\varepsilon}|h(\alpha'_\varepsilon) - h(\alpha')| = \frac{1}{\varepsilon}|h'(\tau_\varepsilon)||\alpha'_\varepsilon - \alpha'|,$$

where $\min(\alpha'_\varepsilon, \alpha') \leq \tau_\varepsilon \leq \max(\alpha'_\varepsilon, \alpha)$. Then by (3.111)₂ we get $\frac{1}{2k} \leq \tau_\varepsilon \leq k+1$ and hence

$$|h'(\tau_\varepsilon)| \leq \max_{\frac{1}{2k} \leq \delta \leq k+1} |h'(\delta)|.$$

Thus by (3.112) we conclude that for $\rho \in (1/k, 1)$

$$\frac{1}{\varepsilon}|h(\alpha'_\varepsilon) - h(\alpha')| \leq C. \quad (3.125)$$

Thus (3.117), (3.120)-(3.125) and the assumptions on the initial iterate (3.82) and (3.102) imply that

$$\frac{1}{\varepsilon} \left| G(\Xi_\varepsilon) + \frac{(v_\varepsilon - v_0)^2}{2} - G(\Xi) - \frac{(v - v_0)^2}{2} \right|$$

is bounded on $(0, 1)$ by an integrable function. Letting $\varepsilon \rightarrow 0$, and using the Dominated Convergence theorem, (A2) and the fact that $(\alpha, \beta, \gamma, v)$ is the minimizer, we obtain the identity (3.119).

Step 3. Conclusion of the computation. The last step is to compute the right-hand side of (3.119). Note first that

$$\begin{aligned} \frac{d\Xi_\varepsilon^1}{d\varepsilon} &= \frac{d}{d\varepsilon} \beta_\varepsilon \rho^{2/3} = 3 \left(\frac{\mu}{\alpha_0^{2/3}} \right)' \rho^{2/3} \\ \frac{d\Xi_\varepsilon^2}{d\varepsilon} &= \frac{d\Xi_\varepsilon^3}{d\varepsilon} = \frac{d}{d\varepsilon} \left(\frac{\alpha_\varepsilon}{\rho} \right) = \frac{3\mu}{\rho} \\ \frac{d\Xi_\varepsilon^4}{d\varepsilon} &= \frac{d}{d\varepsilon} \left(\frac{\gamma_\varepsilon}{\rho^{1/3}} \right) = \frac{2\mu}{\alpha_0^{1/3} \rho^{1/3}} \\ \frac{d\Xi_\varepsilon^5}{d\varepsilon} &= \frac{d\Xi_\varepsilon^6}{d\varepsilon} = \frac{d}{d\varepsilon} \left(\frac{3}{2} \gamma'_\varepsilon \rho^{2/3} \right) = 3 \left(\frac{\mu}{\alpha_0^{1/3}} \right)' \rho^{2/3} \\ \frac{d\Xi_\varepsilon^7}{d\varepsilon} &= \frac{d}{d\varepsilon} (\alpha'_\varepsilon \rho^{2/3}) = 3\mu' \rho^{2/3} \end{aligned}$$

and

$$\frac{dv_\varepsilon}{d\varepsilon} = \frac{\mu}{h\alpha_0^{2/3}}.$$

Then the integrand in (3.119) is expressed by

$$(v - v_0) \frac{dv_\varepsilon}{d\varepsilon} \Big|_{\varepsilon=0} + G_{,i}(\Xi) \frac{d\Xi_\varepsilon^i}{d\varepsilon} \Big|_{\varepsilon=0} = a\mu + b\mu',$$

where

$$\begin{aligned} a(\rho) = & -G_{,1}(\Xi) \frac{2\alpha'_0}{\alpha_0^{5/3}} \rho^{2/3} + G_{,2}(\Xi) \frac{3}{\rho} + G_{,3}(\Xi) \frac{3}{\rho} \\ & + G_{,4}(\Xi) \frac{2}{\alpha_0^{1/3} \rho^{1/3}} - (G_{,5}(\Xi) + G_{,6}(\Xi)) \frac{\alpha'_0}{\alpha_0^{4/3}} \rho^{2/3} + \frac{(v - v_0)}{h\alpha_0^{2/3}} \end{aligned} \quad (3.126)$$

and

$$b(\rho) = \frac{3\rho^{2/3}}{\alpha_0^{2/3}} (G_{,1}(\Xi) + G_{,5}(\Xi)\alpha_0^{1/3} + G_{,6}(\Xi)\alpha_0^{1/3} + G_{,7}(\Xi)\alpha_0^{2/3}). \quad (3.127)$$

Thus by (3.119) we have $(a\mu + b\mu') \in L^1$ and

$$\int_{1/k}^1 (a\mu + b\mu') d\rho = 0. \quad (3.128)$$

Now, we claim $a \in L^1(1/k, 1)$. By (A3) and definition (3.77) of G it follows that for $\rho \in (1/k, 1)$

$$\left| G_{,1}(\Xi) \frac{\alpha'_0}{\alpha_0^{5/3}} \rho^{2/3} \right| \leq \left| \frac{\varphi'(\beta\rho^{2/3})\alpha'_0}{l_k^{5/3}} \right| \leq C (|\beta\rho^{2/3}|^{3p-1} + 1) |\alpha'_0|,$$

$$\frac{1}{\rho} |G_{,2}(\Xi) + G_{,3}(\Xi)| \leq 2k |\psi'(\alpha/\rho)| \leq C (|\alpha/\rho|^{p-1} + 1),$$

and

$$\left| G_{,4}(\Xi) \frac{1}{\alpha_0^{1/3} \rho^{1/3}} \right| \leq \frac{k^{1/3}}{l_k^{1/3}} |g'(\gamma/\rho^{2/3})| \leq C (|\gamma/\rho^{2/3}|^{q-1} + 1).$$

As the right-hand sides of the inequalities above are integrable on $(1/k, 1)$ we have $a \in L^1(1/k, 1)$ and this, in turn, implies $b\mu' \in L^1(1/k, 1)$. Now, we set $z(\rho) = \int_1^\rho a(s) ds$ for $\rho \in (1/k, 1)$. Then z is absolutely continuous and so is μz . Since $(\mu z)|_{\rho=1/k} = (\mu z)|_{\rho=1} = 0$ we get

$$0 = \int_{1/k}^1 (\mu z)' d\rho = \int_{1/k}^1 \left(\mu' \int_1^\rho a(s) ds + \mu a \right) d\rho.$$

Then (3.128) becomes

$$\int_{S_k} \left(- \int_1^\rho a(s) ds + b \right) f d\rho = 0. \quad (3.129)$$

By the properties of f we obtain that there is c_k such that

$$b - \int_1^\rho a(s) ds = c_k \quad \text{a.e. } \rho \in S_k.$$

Since k is arbitrary, the above equality is valid for all $k \in \mathbb{N}$. In this case $S_k \subset S_{k+1}$ implies that $c_k = c_{k+1}$. As $\bigcup_k S_k = \{\rho \in (0, 1) : 0 < \alpha' < \infty\}$ and $m((0, 1) \setminus \bigcup_k S_k) = 0$, we conclude

$$b - \int_1^\rho a(s) ds = \text{const.} \quad \text{a.e. } \rho \in (0, 1). \quad (3.130)$$

Now, let us fix $\delta \in (0, 1)$. By the above argument $a \in L^1(\delta, 1)$ and (3.130) implies $b \in W^{1,1}(\delta, 1)$ with the weak derivative $b' = a$. Moreover, by (3.108) we have $\alpha_0^{2/3} \in W^{1,1}(\delta, 1)$ and hence $b\alpha_0^{2/3} \in W^{1,1}(\delta, 1)$. At this point, we compute

$$D\Omega(\Gamma^0) = \begin{bmatrix} 1 & 0 & 0 & 0 & \alpha_0^{1/3} & \alpha_0^{1/3} & \alpha_0^{2/3} \\ 0 & 3\left(\frac{\alpha_0}{\rho}\right)^{2/3} & 0 & \alpha_0^{1/3} & 0 & \frac{\alpha'_0 \rho}{\alpha_0^{2/3}} & \frac{\alpha'_0 \rho}{\alpha_0^{1/3}} \\ 0 & 0 & 3\left(\frac{\alpha_0}{\rho}\right)^{2/3} & \alpha_0^{1/3} & \frac{\alpha'_0 \rho}{\alpha_0^{2/3}} & 0 & \frac{\alpha'_0 \rho}{\alpha_0^{1/3}} \end{bmatrix}$$

and notice that definitions (3.126) and (3.127) of a and b imply

$$b\alpha_0^{2/3} = 3\rho^{2/3}G_{,i}(\Xi)\Omega_{,1}^i(\Gamma^0) = 3\rho^{2/3}G_1(\rho)$$

while its weak derivative is expressed as

$$\begin{aligned} \frac{d}{d\rho}b\alpha_0^{2/3} &= a\alpha_0^{2/3} + b\frac{2\alpha'_0}{3\alpha_0^{1/3}} = \\ &= \rho^{-1/3}G_{,i}(\Xi) \left(\Omega_{,2}^i(\Gamma^0) + \Omega_{,3}^i(\Gamma^0) \right) + \frac{v - v_0}{h} \\ &= \rho^{-1/3}G_2(\rho) + \frac{v - v_0}{h}. \end{aligned}$$

We conclude that, for $\delta \in (0, 1)$,

$$\rho^{2/3}G_1(\rho) \in W^{1,1}(\delta, 1), \quad \rho^{-1/3}G_2(\rho) \in L^1(\delta, 1), \quad (3.131)$$

and for almost every $\rho \in (0, 1)$

$$3\rho^{2/3}G_1(\rho) = \int_1^\rho \left(s^{-1/3}G_2(s) + \frac{v(s) - v_0(s)}{h} \right) ds + \text{const.} \quad (3.132)$$

Finally, to prove (3.106), we compute

$$\begin{aligned} (\alpha - \alpha_0)' &= h \left(3\alpha_0^{2/3}v \right)' = h \left(\frac{2\alpha'_0}{\alpha_0^{1/3}}v + 3\alpha_0^{2/3}v' \right) \\ &= (\alpha - \alpha_0)\frac{2\alpha'_0}{3\alpha_0} + (\beta - \beta_0)\alpha_0^{2/3} \end{aligned}$$

and hence

$$\alpha' = \frac{\alpha'_0}{3} \left(1 + \frac{2\alpha}{\alpha_0} \right) + (\beta - \beta_0)\alpha_0^{2/3}. \quad (3.133)$$

Similarly,

$$\begin{aligned} (\gamma - \gamma_0)' &= h \left(2\alpha_0^{1/3}v \right)' = \frac{2}{3} \left(\frac{\alpha - \alpha_0}{\alpha_0^{1/3}} \right)' \\ &= \frac{2}{3\alpha_0^{1/3}} \left(\alpha' - \frac{\alpha'_0}{3} \left(2 + \frac{\alpha}{\alpha_0} \right) \right) \end{aligned}$$

and hence

$$\alpha' = \frac{\alpha'_0}{3} \left(2 + \frac{\alpha}{\alpha_0} \right) + \frac{3}{2}(\gamma' - \gamma_0')\alpha_0^{1/3}. \quad (3.134)$$

Now, take $\delta \in (0, 1)$. Then from (3.133) and (3.134) it follows that for all $\rho \in (\delta, 1)$

$$|\alpha'| \leq \frac{|\alpha'_0|}{3} \left(1 + \frac{2\lambda}{\alpha_0(\delta)} \right) + |\beta - \beta_0| \lambda^{2/3} \quad (3.135)$$

and

$$|\alpha'| \leq \frac{|\alpha'_0|}{3} \left(2 + \frac{\lambda}{\alpha_0(\delta)} \right) + \frac{3}{2} |\gamma' - \gamma'_0| \lambda^{1/3}. \quad (3.136)$$

Since δ is arbitrary and $\beta - \beta_0 \in L^{3p}(\delta, 1)$, $\gamma' - \gamma'_0 \in L^q(\delta, 1)$, the assumption (3.102) and last two inequalities imply that for each $\delta \in (0, 1)$

$$\alpha' \in L^{3p}(\delta, 1) \bigcap L^q(\delta, 1). \quad (3.137)$$

This completes the proof. \square

3.3.6 Regularity

First, we claim that for each representative of the minimizer $(\alpha, \beta, \gamma, v) \in \mathcal{A}_\lambda$ in the theorem (3.5) we can alter α' on a set of measure zero such that functions G_1 and G_2 defined in (3.103) and (3.104) satisfy

$$3\rho^{2/3}G_1(\rho) = \int_1^\rho s^{-1/3}G_2(s) + \frac{v(s) - v_0(s)}{h} ds + C_0, \quad \text{for all } \rho \in (0, 1].$$

Indeed, let us fix representatives $(\alpha, \beta, \gamma, v)$ and $(\alpha_0, \beta_0, \gamma_0, v_0)$. Define

$$z(\rho) = \frac{1}{3\rho^{2/3}} \int_1^\rho s^{-1/3}G_2(s) + \frac{v(s) - v_0(s)}{h} ds + C_0 \quad (3.138)$$

and let $A = \{\rho \in (0, 1] : G_1(\rho) \neq z(\rho)\}$. Take any $\rho_0 \in A$ and define

$$y_0 = \left(z(\rho) - \varphi'(\beta\rho^{2/3}) - 2g'(3\gamma'\rho^{1/3})(\alpha_0/\rho)^{1/3} \right) \Big|_{\rho=\rho_0}.$$

Then by (A1) and (A2) it follows that there exists a unique x_0 such that $h'(x_0) = y_0 (\rho_0/\alpha_0(\rho_0))^{2/3}$. Now, by definition of G_1 we have for all $\rho \in (0, 1]$

$$G_1(\rho) = \varphi'(\beta\rho^{2/3}) + 2g'(3\gamma'\rho^{1/3}/2)(\alpha_0/\rho)^{1/3} + h'(\alpha')(\alpha_0/\rho)^{2/3}. \quad (3.139)$$

Thus assigning $\alpha'(\rho_0) = x_0$ we get $G_1(\rho_0) = z(\rho_0)$. In the end, after altering this way α' on the set A , we get that $G_1(\rho) = z(\rho)$ for all $\rho \in (0, 1]$. Moreover by (3.105) we have $mA = 0$ and this finishes the proof.

The following regularity lemma requires a smoother initial iterate than before.

In particular we prove:

Lemma 3.8 (Regularity). *Let $(\alpha, \beta, \gamma, v) \in \mathcal{A}_\lambda$ be the minimizer of I over \mathcal{A}_λ .*

Assume that the initial iterate $(\alpha_0, \beta_0, \gamma_0, v_0)$ satisfies (3.82),

$$\alpha_0, \gamma_0 \in C^1(0, 1] \quad \text{and} \quad \beta_0 \in C(0, 1]. \quad (3.140)$$

Then

$$\alpha, \gamma, v \in C^1(0, 1] \quad \text{and} \quad \beta \in C(0, 1]. \quad (3.141)$$

Proof. Clearly, we can pick a representative $(\alpha, \beta, \gamma, v)$ such that $\alpha, \gamma, v \in C(0, 1]$.

Proceeding as in (3.133) and (3.134), the constraints $\frac{\alpha-\alpha_0}{h} = 3\alpha_0^{2/3}v$, $\frac{\gamma-\gamma_0}{h} = 2\alpha_0^{1/3}v$

and $\frac{\beta-\beta_0}{h} = 3v'$ imply for a.e. $\rho \in (0, 1)$

$$\beta\rho^{2/3} = \alpha'(\rho/\alpha_0)^{2/3} + f_1(\rho) \quad (3.142)$$

and

$$\frac{3}{2}\gamma'\rho^{1/3} = \alpha'(\rho/\alpha_0)^{1/3} + f_2(\rho), \quad (3.143)$$

where

$$\begin{aligned} f_1(\rho) &= \beta_0 \rho^{2/3} - \frac{\alpha'_0 \rho^{2/3}}{3\alpha_0^{2/3}} \left(1 + \frac{2\alpha}{\alpha_0}\right) \\ f_2(\rho) &= \frac{3}{2} \gamma'_0 \rho^{1/3} - \frac{\rho^{1/3}}{\alpha_0^{1/3}} \left(2 + \frac{\alpha}{\alpha_0}\right). \end{aligned}$$

We note that (3.140) implies that f_1 and f_2 are continuous on $(0, 1]$ functions.

First, we alter β and γ' so that equality in (3.142) and (3.143) holds for all $\rho \in (0, 1)$. Hence by (3.139) we have for all $\rho \in (0, 1]$

$$\begin{aligned} G_1(\rho) &= \varphi'(\alpha'(\rho/\alpha_0)^{2/3} + f_2(\rho)) \\ &\quad + 2g'(\alpha'(\rho/\alpha_0)^{1/3} + f_1(\rho))(\alpha_0/\rho)^{1/3} \\ &\quad + h'(\alpha')(\alpha_0/\rho)^{2/3}. \end{aligned} \tag{3.144}$$

and this suggests to define $f : \mathbb{R}_+ \times (0, 1] \rightarrow \mathbb{R}$ by

$$\begin{aligned} f(x, \rho) &= \varphi'(x(\rho/\alpha_0)^{2/3} + f_2(\rho)) \\ &\quad + 2g'(x(\rho/\alpha_0)^{1/3} + f_1(\rho))(\alpha_0/\rho)^{1/3} \\ &\quad + h'(x)(\alpha_0/\rho)^{2/3}. \end{aligned} \tag{3.145}$$

Now, define $A = \{\rho \in (0, 1] : G_1(\rho) \neq z(\rho)\}$. Clearly, $mA = 0$ and note that from (3.144) it follows

$$G_1(\rho) = f(\alpha', \rho) = z(\rho), \quad \rho \notin A. \tag{3.146}$$

Take $\rho_0 \in A$. Then, as $\rho_0 > 0$ and $\alpha_0(\rho_0) > 0$, properties (A1)-(A3) imply that $f_x(x, \rho_0) > 0$ for all $x \in \mathbb{R}_+$; moreover, $\lim_{x \rightarrow 0+} f(x, \rho_0) = -\infty$ and $\lim_{x \rightarrow +\infty} f(x, \rho_0) = +\infty$. Hence there exists unique $x_0 \in \mathbb{R}_+$ such that $f(x_0, \rho_0) = z(\rho_0)$.

At this point we are ready to assign new values for α', β and γ' . Define

$$\alpha'(\rho_0) = x_0, \quad \beta(\rho_0) = \frac{x_0}{\alpha_0(\rho_0)^{2/3}} + \frac{f_1(\rho_0)}{\rho_0^{2/3}}$$

and

$$\gamma'(\rho_0) = \frac{2}{3} \left(\frac{x_0}{\alpha_0(\rho_0)^{1/3}} + \frac{f_2(\rho_0)}{\rho_0^{1/3}} \right).$$

This implies that (3.142) and (3.143) hold at $\rho = \rho_0$ and hence by (3.139)

$$G_1(\rho_0) = f(x_0, \rho_0) = f(\alpha'(\rho_0), \rho_0) = z(\rho_0). \quad (3.147)$$

As $\rho_0 \in A$ was arbitrary (3.146) and (3.147) imply

$$G_1(\rho) = f(\alpha', \rho) = z(\rho), \quad \rho \in (0, 1]. \quad (3.148)$$

Hence G_1 is continuous on $(0, 1]$ and therefore $\alpha' > 0$ for all $\rho \in (0, 1]$.

Now, let us assume $\rho_k \rightarrow \rho_0$ and $\alpha'(\rho_k) \rightarrow l \in [0, \infty]$ with $\rho_k, \rho_0 \in (0, 1]$, $k \in \mathbb{N}$.

First, we claim that $l \in (0, \infty)$. Indeed, assume that $l = 0$ or $l = +\infty$. Then by continuity of α_0 we have $\alpha_0(\rho_k) \rightarrow \alpha_0(\rho_0) > 0$ and hence properties (A1)-(A3), together with continuity of f_1 and f_2 , imply $\lim_{k \rightarrow \infty} f(\alpha'(\rho_k), \rho_k) = \mp\infty$ respectively.

Thus by continuity of G_1 and (3.148) we have

$$G_1(\rho_0) = \lim_{k \rightarrow \infty} G_1(\rho_k) = \lim_{k \rightarrow \infty} f(\alpha'(\rho_k), \rho_k) = \mp\infty \quad (3.149)$$

which is a contradiction. Therefore we assume $l \in (0, \infty)$. As f_1, f_2 are continuous on $(0, 1]$, we must have $\lim_{k \rightarrow \infty} f(\alpha'(\rho_k), \rho_k) = f(l, \rho_0)$ and therefore by (3.148) we get

$$\begin{aligned} f(\alpha'(\rho_0), \rho_0) &= G_1(\rho_0) = \lim_{k \rightarrow \infty} G_1(\rho_k) \\ &= \lim_{k \rightarrow \infty} f(\alpha'(\rho_k), \rho_k) = f(l, \rho_0). \end{aligned} \quad (3.150)$$

By the strict monotonicity of $f(\cdot, \rho_0)$ we get $\alpha_0(\rho_0) = l$ and conclude that α' is continuous on $(0, 1]$.

Finally, from the discussion above it follows that equalities (3.142) and (3.143) hold for all $\rho \in (0, 1]$. The continuity of f_1, f_2 and α' imply $\beta, \gamma' \in C(0, 1]$. Moreover, as $\frac{\alpha - \alpha_0}{h} = 3\alpha_0^{2/3}v$ for all $\rho \in (0, 1]$, we obtain $v \in C^1(0, 1]$. This finishes the proof. \square

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